CS 140 Algorithms – Fall 2013

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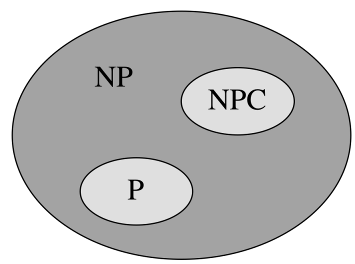
Lecture 21. NP-Completeness Reductions

1 Important Terminology and Concepts.

- Tractability:

- Intractability:

- NP:



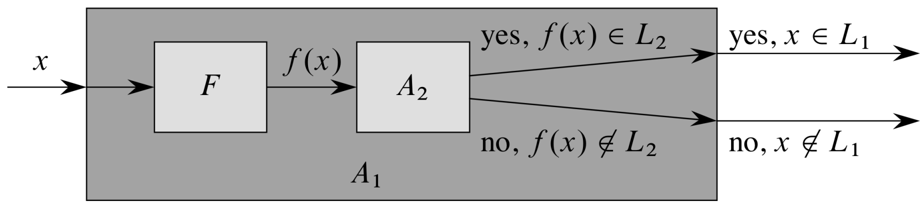
- P:

- NP Hard:

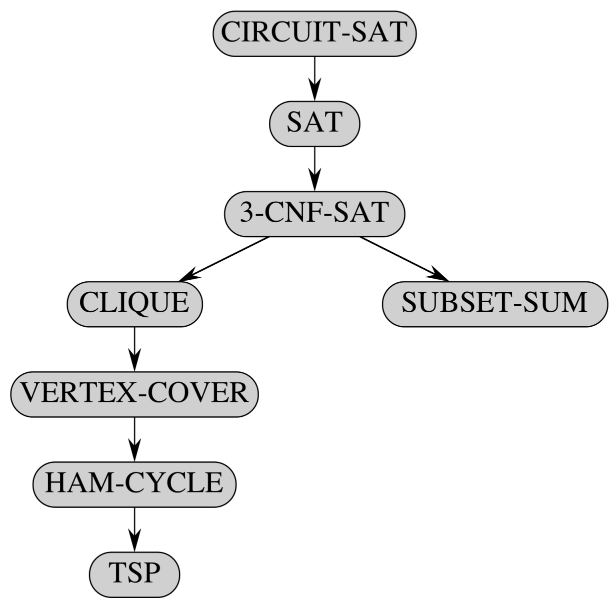
- NP Complete:

- Optimization vs. Decision (e.g., MST vs. MST-Decision):

- Reduction Function ($\leq_P$):



- Cook-Levin Theorem: *Every decision problem in NP can be (quickly) converted into a corresponding 3-SAT decision problem.*



- Proving NP-Completness:

- Cool resource! `https://complexityzoo.uwaterloo.ca`

2 NP-Complete Problems

2.1 Clique

A ***clique*** in an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ of the vertices, each pair of which is connected by an edge in E. In other words, a clique is a complete subgraph of G. The ***size*** of a clique is the number of vertices it contains. The ***clique problem*** is the optimization problem of finding a clique of maximum size in a graph. As a decision problem, we ask simply whether a clique of a given size k exists in the graph. Formally:

CLIQUE $= \{\langle G, k\rangle : G$ is a graph containing a clique of size $k\}$.

Last time we showed in class that **Clique is NP-Complete**.

2.2 Vertex Cover

A ***vertex cover*** of an undirected graph $G = (V, E)$ is a subset $V' \subseteq V$ such that if $(u, v) \in E$, then $u \in V'$ or $v \in V'$ (or both). That is, each vertex "covers" its incident edges, and a vertex cover for G is a set of vertices that covers all the edges in E. The ***size*** of a vertex cover is the number of vertices in it. The ***vertex-cover problem*** is to find a vertex cover of minimum size in a given graph. As a decision problem, the goal is to determine whether a graph has a vertex cover of a given size k. Formally:

VERTEX-COVER $= \{\langle G, k\rangle :$ graph G has a vertex cover of size $k\}$.

2.3 Hamiltonian Cycles

A ***hamiltonian cycle*** of an undirected graph $G = (V, E)$ is a simple cycle that contains each vertex in V. A graph that contains a hamiltonian cycle is said to be ***hamiltonian***; otherwise, it is ***nonhamiltonian***. The ***hamiltonian-cycle problem*** is expressed as "Does a graph G have a hamiltonian cycle", or formally:

HAM-CYCLE $= \{\langle G\rangle : G$ is a hamiltonian graph$\}$.

2.4 Traveling-Salesperson Problem

In the ***traveling-salesperson problem***, which is closely related to the hamiltonian-cycle problem, a salesperson must visit n cities. Modeling the problem as a complete graph with n vertices, we can say that the salesperson wishes to make a ***tour***, or hamiltonian cycle, visiting each city exactly once and finishing at the city he or she starts from. The salesperson incurs a nonnegative interger cost $c(i, j)$ to travel from city i to city j, and the salesperson wishes to make a tour who total cost is minimum, where total cost is the sum of the individual costs along along the edges of the tour. Formally, as a decision problem:

TSP $= \{\langle G, c, j\rangle : G$ has a tsp tour with cost at most $k\}$.

3 Vertex Cover

Theorem 34.12 The vertex-cover problem is NP-Complete.

3.1 Vertex cover is in NP

A) V-C Verification Algorithm:

B) V-C Verification Runtime:

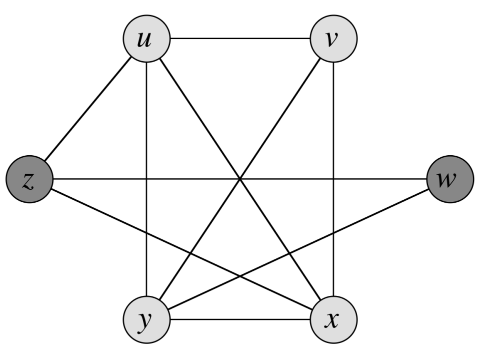
Given a graph $G = (V, E)$, its ***complement*** is $\bar{G} = (V, \bar{E})$, where $\bar{E} = \{(u, v) : u, v \in V; u \neq v; (u, v) \notin E\}$.

3.2 Vertex cover is NP-hard

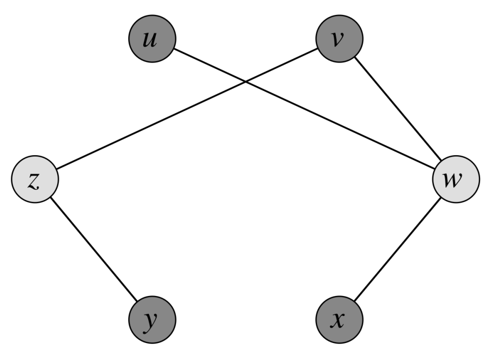
Here we show that the vertex-cover problem is NP-hard by showing that CLIQUE $\leq_P$ VERTEX-COVER.

A) Reduction Algorithm:

B) Reduction Runtime:



(a)



(b)

C) G has a clique of size k iff the graph $\bar{G}$ has a vertex cover of size $|V| - k$.

- IFF $\Rightarrow$ (yes$\rightarrow$yes):

- IFF $\Leftarrow$ (no$\rightarrow$no):

Conclusion:

4 SAT

Worksheet SAT is NP-Complete.

4.1 SAT is in NP

A) SAT Verification Algorithm:

B) SAT Verification Runtime:

4.2 SAT is NP-hard

Here, you should show that SAT is NP-hard by showing that 3-SAT $\leq_P$ SAT.

A) Reduction Algorithm:

B) Reduction Runtime:

C) 3-SAT is satisfied iff the corresponding instance of SAT is.

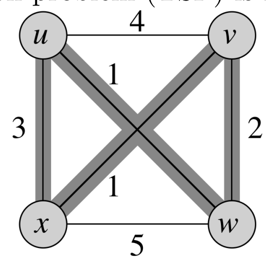
- IFF $\Rightarrow$ (yes$\rightarrow$yes):

- IFF $\Leftarrow$ (no$\rightarrow$no):

Conclusion:

5 Traveling Salesperson

Theorem 34.14 The traveling salesperson problem (TSP) is NP-Complete.



5.1 TSP is in NP

A) TSP Verification Algorithm:

B) TSP Verification Runtime:

5.2 TSP is NP-hard

Here we show that the TSP is NP-hard by showing that HAM-CYCLE $\leq_P$ TSP.

A) Reduction Algorithm:

B) Reduction Runtime:

C) G has a hamiltonian cycle iff G' has a tour of cost at most 0.

- IFF $\Rightarrow$ (yes$\rightarrow$yes):

- IFF $\Leftarrow$ (no$\rightarrow$no):

Conclusion:

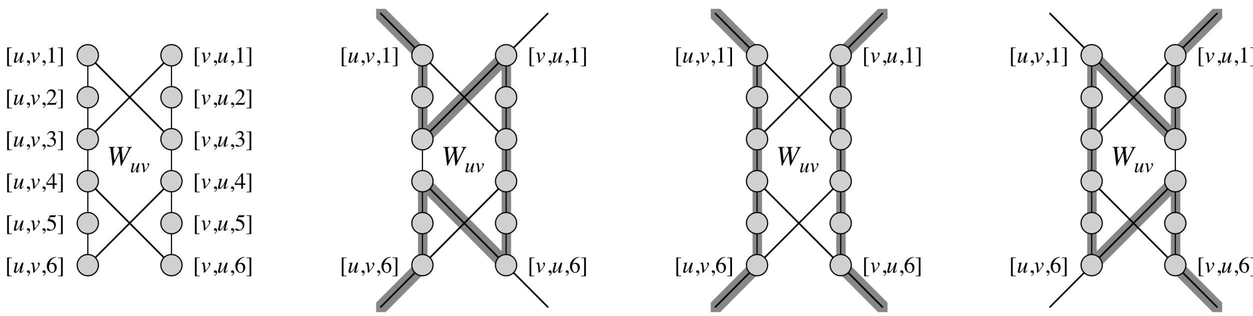
6 Hamiltonian Cycle

Theorem 34.13 The hamiltonian cycle problem is NP-Complete.

6.1 Hamiltonian cycle is in NP

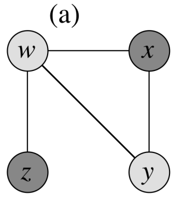
A) H-C Verification Algorithm:
See lecture notes from last class.

B) H-C Verification Runtime:
See lecture notes from last class.

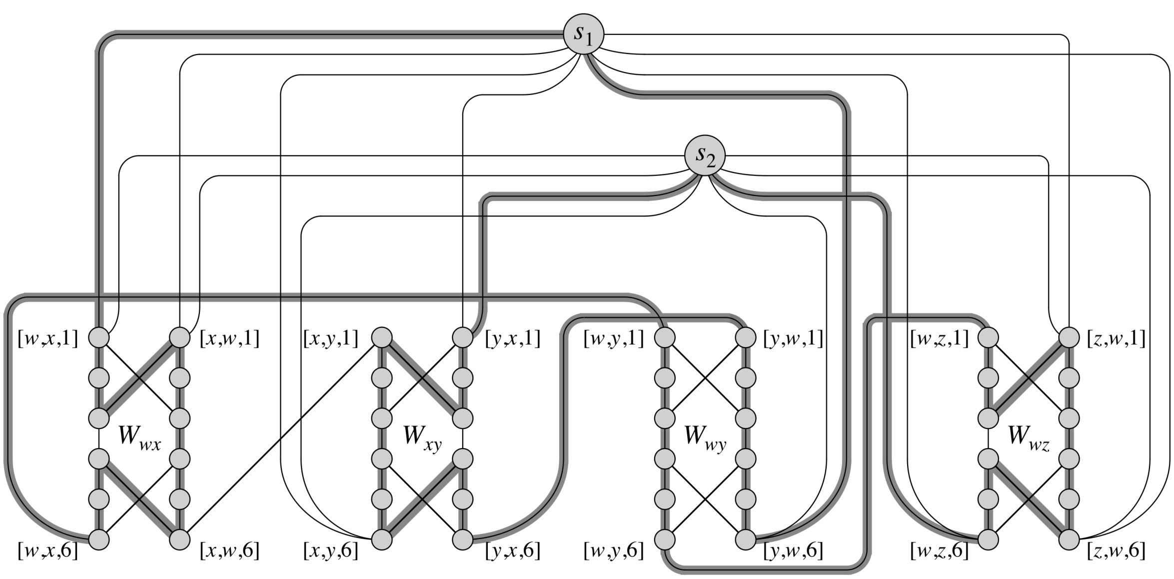


(a) (b) (c) (d)

(a)



(b)



6.2 Hamiltonian cycle is NP-hard

Here we show that the hamiltonian cycle problem is NP-hard by showing that VERTEX-COVER $\leq_P$ HAM-CYCLE.

A) Reduction Algorithm:

B) Reduction Runtime:

C) G has a vertex cover of size k iff G' has a hamiltonian cycle.

- IFF $\Rightarrow$ (yes$\rightarrow$yes):

- IFF $\Leftarrow$ (no$\rightarrow$no):

Conclusion: