Product spaces

Consider two Hilbert spaces $U \otimes V$ with the basis vectors $\{|u_i\rangle\}$ and $\{|v_i\rangle\}$, the product space is obtained by considering products of the form

$$|u\rangle\,|v\rangle$$

$$|u\rangle = \sum_i \alpha_i |u_i\rangle$$

$$|v\rangle = \sum_j \beta_j |v_j\rangle$$

$$\Rightarrow |u\rangle|v\rangle = \sum_{ij} \alpha_i \beta_j |u_i\rangle|v_j\rangle$$

$|u_i\rangle|v_j\rangle$ form a basis for the product space

$\dim(U) = n$, $\dim(V) = m \Rightarrow \dim(U \otimes V) = nm$

$U \otimes V$ is the product space.

Operators in the product space

$$O = \sum_{ijkl} O_{ij,kl} |u_i\rangle|v_j\rangle\langle u_k|\langle v_l|$$

Matrices in nm dimensional space.

Examples Two spin $1/2$'s in quantum mechanics

U spanned by $|\uparrow\rangle_1$ and $|\downarrow\rangle_1$

V " " $|\uparrow\rangle_2$ and $|\downarrow\rangle_2$

$U \otimes V$ spanned by $|\uparrow\rangle_1|\uparrow\rangle_2$, $|\uparrow\rangle_1|\downarrow\rangle_2$, $|\downarrow\rangle_1|\uparrow\rangle_2$, $|\downarrow\rangle_1|\downarrow\rangle_2$

Example singlet

$$|s\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

Triplet states

$$|t1\rangle = |\uparrow\rangle_1 |\uparrow\rangle_2$$

$$|t2\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

$$|t3\rangle = |\downarrow\rangle_1 |\downarrow\rangle_2$$

Operators

$S_{x1}, S_{y1}, S_{z1}, I_1$ } Identity

$S_{x2}, S_{y2}, S_{z2}, I_2$

$$S_x = S_{x1} + S_{x2} = S_{x1} I_2 + S_{x2} I_2$$

↓

4x4 matrix

Show that

$$[S_x, S_y] = i\hbar S_z$$

and cyclic permutations

Partial traces

Consider an operator in $U \otimes V$, which has the form

$$H = \sum_{ij,kl} H_{ij,kl} |u_i\rangle |v_j\rangle \langle u_k| \langle v_l|$$

The trace of H is

$$Tr(H) = \sum_{m,n} \langle u_m | \langle v_n | \ H \ | v_n \rangle | u_m \rangle$$

$$= \sum_{m,n} H_{mn,mn}$$

Partial trace - Trace over the basis vectors of only one space

$$Tr_u(H) = \sum_m \langle u_m | H | u_m \rangle$$

$$= \sum_{m,j,l} H_{mj,ml} |v_j\rangle \langle v_l|$$

$Tr_u(H)$ is an operator in V

Illy $Tr_v(H)$ is an operator in U

Density matrices in quantum mechanics

External probabilities p_i for states $|\psi_i\rangle$

Expectation value of a Hermitian operator A

$$\langle A \rangle = \sum_i p_i \langle \psi_i | A | \psi_i \rangle$$

$\{|\psi_i\rangle\}$ need not be orthogonal or even span the vector space

$$|\varphi_i\rangle = \sum_j \alpha_{ij} |v_j\rangle \quad \{|v_j\rangle\} \text{ is an orthonormal basis}$$

$$\cancel{\langle A\rangle = \sum_{ijk} p_i \alpha_{ij}^* \langle v_j | A}$$

$$\langle A\rangle = \sum_{ijk} p_i \alpha_{ij}^* \alpha_{ik} \langle v_j | A | v_k\rangle$$

$$= \mathrm{Tr}(\rho A)$$

$$\rho = \sum_i p_i \alpha_{ij}^* \alpha_{ik}$$

Operator $\rho = \sum_i p_i |\varphi_i\rangle\langle\varphi_i|$

$$= \sum_{ijk} p_i \alpha_{ij}^* \alpha_{ik} |v_k\rangle\langle v_j|$$

Prove that ρ is Hermitian & $\mathrm{Tr}(\rho) = 1$

Examples:

Canonical ensemble $\rho = e^{-H/k_BT}$

$$\rho = \sum_i e^{-E_i/k_BT} |\varphi_i\rangle\langle\varphi|_i$$

Energy eigenstates

Eg A pure state

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{in some basis}$$

Otherwise mixed state.

Entropy $S = -k_B \, Tr(\rho \ln \rho)$ ~~→ mp~~

~~For a pure state $S = 0$~~

For a pure state $S = 0$.

Entanglement

All states in a product space are not product states.

eg Singlet state $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$

$$\neq \left(a_1|\uparrow\rangle_1 + b_1|\downarrow\rangle_1\right)\left(a_2|\uparrow\rangle_2 + b_2|\downarrow\rangle_2\right)$$

for any a_1, b_1, a_2 & b_2

Entanglement is crucial for quantum computing.

~~How d~~

Unentangled state (Product state)

$$|\psi\rangle = \frac{1}{2}\left(|\uparrow\rangle_1 |\uparrow\rangle_2 + |\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2 + |\downarrow\rangle_1 |\downarrow\rangle_2\right)$$

$$= \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 + |\downarrow\rangle_1\right)\left(|\uparrow\rangle_2 + |\downarrow\rangle_2\right)$$

How do we know if a given state is entangled or not?

Schmidt decomposition

Let H_1 and H_2 be two Hilbert spaces. A general state in $H_1 \otimes H_2$

$$|\psi\rangle = \sum_{i,j} a_{ij} |h_i\rangle_1 |g_j\rangle_2$$

$\dim(H_1)$ need not be equal to $\dim(H_2)$

Given $|\psi\rangle$, $\exists$ orthonormal bases ~~$|h_i\rangle$ and $|g_i\rangle$~~
$|h'\rangle$ and $|g'\rangle$: ...

$|\psi\rangle = \sum_i \lambda_i |h'_i\rangle |g'_i\rangle$ $\quad$ i = smaller of dim (H) or dim (G)

Schmidt decomposition

$\lambda_i > 0$ & $\sum_i \lambda_i^2 = 1$ if $|\psi\rangle$ is normalized

Product state iff $\lambda_j = 1$ for some j, otherwise entangled state.

If a_{ij} is the matrix A ($m \times n$ matrix),

Let $\dim(G) = n \le \dim(H) = m$

$A = U D V^*$ (Singular value decomposition)

U: $m \times m$ unitary

D: $m \times n$ rectangular diagonal matrix

V: $n \times n$ unitary matrix

Diagonalization is a special case for square matrices

Quantifying entanglement

Reduced density matrix ρ_A

Construct density matrix
$\rho = |\psi\rangle\langle\psi|$, where $|\psi\rangle = \sum_{ij} a_{ij} |h_i\rangle |g_j\rangle$
$= \sum_i \lambda_i |h_i\rangle |g_i\rangle$

$\rho_A = Tr_B(\rho)$

$|\psi\rangle = \sum_{ij} a_{ij} |h_i\rangle |g_j\rangle$

$\rho = |\psi\rangle\langle\psi| = \sum_{ijkl} a_{ij} a^*_{kl} |h_i\rangle |g_j\rangle \langle h_k| \langle g_l|$

$$\rho_A = Tr_B(\rho) = \sum_{ijk} a_{ij} a^*_{kj} |h_i\rangle \langle h_k|$$

$$= \sum_i \lambda_i^2 |h_i\rangle \langle h_i|$$

$$\rho = Tr_A(\rho_A) = Tr_{AB}(\rho) = 1 \quad \text{(Verify)}$$

Entanglement entropy

$S = -k_B Tr(\rho_A \ln \rho_A) = -k_B Tr(\rho_B \ln \rho_B)$

follows from the Schmidt decomposition

$S = 0$ for product or unentangled state

Larger S, greater the entanglement

Singlet

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

$S = k_B \ln 2$

Schrödinger cat state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2\right)$$

$S = k_B \ln 2$

$|\uparrow\rangle_1 |\uparrow\rangle_2 \quad S = 0$

$|\uparrow\rangle_1 |\downarrow\rangle_2 \quad S = 0$

$|\downarrow\rangle_1 |\downarrow\rangle_2 \quad S = 0$

λ_i^2 are eigenvalues of ρ_A (or ρ_B)

Proof of Schmidt decomposition

$$|\psi\rangle = \sum_{ij} a_{ij} |h_i\rangle |g_j\rangle = \sum_i \Big(|h_i\rangle \sum_j a_{ij} |g_j\rangle\Big)$$

$$= \sum_i |h_i\rangle |\tilde{h}_i\rangle \quad \text{, where } |\tilde{h}_i\rangle = \sum_j a_{ij} |g_j\rangle$$

Note $|\tilde{h}_i\rangle$ are not necessarily orthonormal

Choose $|h_i\rangle$ to be the eigenvectors of ρ_A

$$\rho_A = \sum_i p_i |h_i'\rangle\langle h_i'|$$

$$\rho_A = Tr_B(|\psi\rangle\langle\psi|) = Tr_B\Big(\sum_{ij} |h_i'\rangle |\tilde{h}_i\rangle \langle\tilde{h}_j| \langle h_j'|\Big)$$

$$= \sum_{ij} \Big(|h_i'\rangle\langle h_j'|\Big) Tr_B\Big(|\tilde{h}_i\rangle\langle\tilde{h}_j|\Big)$$

$$= \sum_{ijm} |h_i'\rangle\langle h_j'| \Big(\langle m|\tilde{h}_i\rangle\langle\tilde{h}_j|m\rangle\Big)$$

$|m\rangle$ is an orthonormal basis for B with vector space G.

$$= \sum_{ij} |h_i'\rangle\langle h_j'| \sum_m \langle\tilde{h}_j|m\rangle\langle m|\tilde{h}_i\rangle$$

$$= \sum_{ij} \langle\tilde{h}_j|\tilde{h}_i\rangle \Big(|h_i'\rangle\langle h_j'|\Big) = \sum_i p_i |h_i'\rangle\langle h_i'|$$

$\Rightarrow \langle\tilde{h}_j|\tilde{h}_i\rangle = p_i\delta_{ij}$ or $\{|\tilde{h}_i\rangle\}$ are orthogonal and can be normalized by

choosing $|g_i'\rangle = \dfrac{|\tilde{h}_i\rangle}{\sqrt{p_i}}$

$\Rightarrow |\psi\rangle = \sum \sqrt{p_i}\, |h_i'\rangle |g_i'\rangle = \sum_i \lambda_i |h_i'\rangle |g_i'\rangle$ where $\lambda_i = \sqrt{p_i}$