PH 205: Mathematical methods of physics

Problem Set 5

1. Consider an AC voltage of the form $V(t) = V_0 \cos \omega t$ coming out of a transformer. This voltage is rectified and used to power a load of resistance R. For all the parts of the problem, assume that the diodes in the rectifier are ideal (i.e. in reverse bias they allow no current flow and in forward bias, have zero resistance and zero voltage drop).

   (a) For a half wave rectifier, calculate the Fourier series for the voltage across the load.

   (b) For a full wave rectifier (assume the transformer has a centre tap), calculate the Fourier series for the voltage across the load.

   (c) Assume that the full wave rectifier has a capacitive filter, i.e. a capacitor of capacitance C is connected across the load. Calculate the Fourier series across the load in this case. Show that this series is a function of a "time factor" that is a solution of a transcendental equation involving the parameters R, C and ω. What do you expect to happen to the coefficients of the Fourier series as C increases?

2. It can be shown that

$$\int_{-\infty}^{\infty} \exp[-\alpha(x - iu)^2]\ dx = \sqrt{\frac{\pi}{\alpha}},$$

   where $\alpha > 0$ and u is a real number. You will learn how to do this when you study complex integration but for the time being you only require the result.

   (a) Show that Fourier transform of a Gaussian is a Gaussian. Specifically show that the Fourier transform of the normalized Gaussian function

$$f(x) = \frac{1}{\pi^{1/4}} e^{-x^2/2},$$

   is also a normalized Gaussian function. Normalized here means that $\int_{-\infty}^{\infty} [f(x)]^2\ dx = 1$.

   (b) Now, show that the Fourier transforms of the normalized Hermite functions (which are the energy eigenfunctions of the one dimensional Harmonic oscillator) are also normalized Hermite functions. The n^{th} order normalized Hermite function is given by

$$f_n(x) = \left(\frac{1}{\sqrt{\pi} 2^n n!}\right)^{1/2} e^{-x^2/2} H_n(x),$$

   where $H_n(x)$ is the n^{th} order Hermite polynomial. The simplest way to prove this is using mathematical induction and the recursion relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

   (c) Why are the results of parts (a) and (b) not unexpected? (*Hint: Go back and look at problem 3(c) of problem set # 4.*)

3. The Dirac delta function $\delta(x)$ can be defined by the relation

$$\int_a^b f(x)\delta(x - x_0)\ dx = \begin{cases} = \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right] & x_0 \in (a, b) \\ = \frac{f(x_0^+)}{2} & x = b \\ = \frac{f(x_0^-)}{2} & x = a \\ = 0 & x_0 \notin (a, b) \end{cases}$$

   where $f(x)$ is any bounded function that is zero everywhere outside some finite interval (which does not have to be the same as (a, b)) and is discontinuous only at a finite number of points.

(a) Using the above definition, prove the following properties of the Dirac delta function in the interval

i. $\delta(x - x_0) = 0, \forall x \neq x_0$

ii. $\int_a^b f(x)\delta'(x - x_0)\ dx = -\int_a^b f'(x)\delta(x - x_0)\ dx$ for $x_0 \in (a, b)$ and if $f(x)$ is differentiable in (a, b)

iii. $\delta[\alpha(x - x_0)] = \frac{1}{|\alpha|}\delta(x - x_0)$ for $\alpha \in \mathcal{R}$

iv. $\delta[g(x)] = \sum_i \delta(x - x_i)/|g'(x_i)|$, where x_i are the zeros of $g(x)$.

(b) Show that the following functions $d(x - x_0, \epsilon)$ in the limit $\epsilon \to 0$ yield the Dirac delta function

i. $\frac{1}{2\epsilon}e^{-|x-x_0|/\epsilon}$

ii. $\frac{\epsilon}{\pi[(x-x_0)^2+\epsilon^2])}$

iii. $\frac{1}{2\sqrt{\pi\epsilon}}\exp(-\frac{(x-x_0)^2}{4\epsilon})$

All you need to show is that

$$\lim_{\epsilon \to 0} d(x - x_0, \epsilon) = 0, \forall x \neq x_0,$$

and

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d(x - x_0, \epsilon)\ dx = 1.$$

(c) Finally show that the function $\frac{\sin[(x-x_0)/\epsilon]}{\pi(x-x_0)}$ also yields the Dirac delta function in the limit $\epsilon \to 0$. You will learn to integrate this function when you study complex integration but for the time being simply use the definition of the delta function in terms of Fourier transfroms.

4. The equation of motion of a forced damped harmonic oscillator in 1D is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t),$$

where m, b and k are the mass, damping constant and spring constant of the oscillator respectively and $F(t)$ is the driving force. The general solution of this equation is $x(t) = X(t) + x_0(t)$, where $x_0(t)$ is a piece which satifies the homogeneous part of the equation of motion (i.e. with the RHS=0). You will calculate only the other piece $X(t)$ in this problem.

(a) What is $X(t)$ for $F(t) = F_0 \cos\omega t$, for some angular frequency ω?

(b) Now, suppose that $F(t)$ is any periodic force with time period $T = 2\pi/\omega$, so that it has a Fourier expansion of the usual form

$$F(t) = \frac{f_0}{2} + \sum_{n=1}^{\infty} [f_n \cos(n\omega t) + \phi_n \sin(n\omega t)]\,.$$

Calculate $X(t)$.

(c) Now, suppose $T \to \infty$. Using the technique of converting Fourier series into Fourier transforms (i.e. sums into integrals) developed in class, write down $X(t)$ as an integral involving $\tilde{F}(\omega)$, the Fourier transform of $F(t)$. Show that this solution is of the form

$$X(t) = \int_{-\infty}^{\infty} G(t - t')F(t')\ dt'.$$

What is the expression for $G(t - t')$ as an integral over ω?

(d) On physical grounds what do you expect the value of $G(t - t')$ to be for $t < t'$ and why?

5. The Coulomb potential of a point charge in d dimensions is described by the usual Poisson equation

$$\nabla^2 V(\mathbf{r}) = -\delta(\mathbf{r} - \mathbf{r}_0),$$

where $\mathbf{r}$ is a d dimensional position vector and $\mathbf{r}_0$ is the location of the point charge. The charge has been assumed to be of unit strength and the permittivity of space taken equal to one for simplicity. For $d = 3$, this yields the solution $V(\mathbf{r}) \sim 1/|\mathbf{r} - \mathbf{r}_0|$. In this problem you will calculate dependence of the potential on $|\mathbf{r} - \mathbf{r}_0|$ for $d > 3$.

(a) Write down the expression for $V(\mathbf{r})$ as in terms of a suitable d-dimensional $\mathbf{k}$ space integral over $\tilde{V}(\mathbf{k})$, its Fourier transform.

(b) The best way to evaluate the above integral is in spherical polar coordinates in d-dimensional $\mathbf{k}$ space. The "z" axis of this space can be taken to be along the direction of $\mathbf{r} - \mathbf{r}_0$ without loss of generality. The volume element

$$d^d\mathbf{k} = k^{d-1} dk d\Omega_d,$$

where $k = |\mathbf{k}|$ and $d\Omega_d$ is the infinitesimal d-dimensional solid angle. Argue that the integral of part (a) gives you an expression of the form

$$V(\mathbf{r}) = \frac{A_d}{|\mathbf{r} - \mathbf{r}_0|^{d-2}},$$

where A_d is a number that can be expressed as a suitable integral.

(c) As an aside, calculate $\int d\Omega_d$ over the surface of an infinite d-dimensional sphere (calculating it over the surface of any closed volume in d-dimensional space will also give the same answer) since it will be useful later in the course. Do this in the following way: Consider

$$\int e^{-r^2} d^d\mathbf{r} = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \exp(-\sum_{i=1}^{d} x_i^2)\ dx_1 \ldots d_{x_d} = \prod_{i=1}^{d} \int_{-\infty}^{\infty} e^{-x_i^2} dx_i,$$

where the first integral is over all d-dimensional space and use the expression for the volume element $d^d\mathbf{r} = r^{d-1} dr d\Omega_d$. Verify that you get $\int d\Omega_3 = 4\pi$.