PH 205: Mathematical methods of physics

Problem Set 7

1. In this problem you will evaluate the Gaussian integral with complex parameters.

   (a) First evaluate

   $$\int_{-\infty}^{\infty} \exp\left[-\alpha(x-u)^2\right]\, dx,$$

   $\alpha > 0$ and u is any complex number. Do this by converting the above integral to a complex integral along an appropriate contour and using the known result for

   $$\int_{-\infty}^{\infty} \exp\left[-\alpha(x-x_0)^2\right]\, dx,$$

   where x_0 is any real number.

   (b) Now, evaluate

   $$\int_{-\infty}^{\infty} \exp\left[-\gamma x^2\right]\, dx,$$

   where γ is a complex number. One again choose an appropriate contour and use the known result for the Gaussian integral. What are the constraints on γ for the above integral to be finite.

2. The Kramers-Kronig relations relate the real and imaginary parts of susceptibilities (also called response functions) in physics. Such a susceptibility $\chi(\omega)$ is a function of angular frequency ω and is of the form

   $$\chi(\omega) = \chi_1(\omega) + \chi_2(\omega),$$

   where $\chi_1(\omega)$ and $\chi_2(\omega)$ are the real and imaginary parts of the susceptibility. $\chi(\omega)$ also has the property that it is analytic in the upper half plane and goes to zero faster than $1/|\omega|$ as $\omega \to \infty$.

   (a) Show that

   $$\chi_1(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega' - \omega} d\omega',$$

   $$\chi_2(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega' - \omega} d\omega',$$

   (b) The linear response of a time dependent quantity $X(t)$ to a time dependent perturbation $F(t)$ is given by

   $$X(t) = \int_{-\infty}^{\infty} G(t-t')F(t')dt',$$

   where $G(t-t')$ is the Green's function which you have encountered before in problem 4 of problem set 5. In that problem you argued that $G(\tau) = 0$ for $\tau < 0'$. The susceptibility $\chi(\omega)$ is the fourier transform of the Green's function and the above fact can be used to argue that $\chi(\omega)$ is analytic in the upper half plane. Further, it can also be argued that for physical systems $\chi(\omega)$ does indeed fall off faster than $1/|\omega|$ as $\omega \to \infty$. $X(t)$ and $F(t)$ are real for for all t for physical systems and this implies that the Green's function is also a real-valued function. Use these facts to convert the integrals of part (a) to those from 0 to ∞.

3. In this problem you will calculate Green's functions (also called propagators) for the following operators, which are very important in physics, using the technique of Fourier transforms and contour integration:

(a) The Helmholtz operator

$$\nabla^2 + m^2,$$

where m is a real number. Calculate the Green's function $G(\mathbf{r}-\mathbf{r}')$ in 3 dimensions.

(b) The D'Alembertian operator

$$\frac{1}{c}\frac{\partial^2}{\partial t^2} - \nabla^2.$$

Here t is time, c, the speed of light and ∇^2 is the usual Laplacian operator in three dimensions. Calculate the Green's function $G(t-t';\mathbf{r}-\mathbf{r}')$.

(c) The Schrödinger operator for a free particle in one dimension

$$i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2}\frac{\partial^2}{\partial x^2}.$$

Calculate the Green's function $G(t-t';x-x')$

(d) The diffusion operator in one dimension

$$\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2},$$

where $D > 0$ and is the diffusion constant. Calculate the Green's function $G(t-t;,x-x')$

4. A very important technique in quantum statistical mechanics is that of Matsubara summation. In this problem you will perform such a sum using the techniques of contour integration. A typical Matsubara sum, which occurs in the physics of solids involving interactions between electrons and phonons at a temperature T is of the form

$$S = -\frac{1}{k_B T}\sum_n f(i\omega_n).$$

Here k_B is the Boltzmann constant and $\omega_n = 2\pi n k_B T/\hbar$ where n is an integer, where $\hbar$ is Planck's constant divided by 2π. The ω_n's are called Matsubara frequencies and $f(i\omega_n)$ is some (not necessarily analytic) function of ω_n. The summation is over all integers n.

(a) Show that the above sum can be evaluated by performing the integral

$$I = \oint_C \frac{dz}{2\pi i} f(z) n(z),$$

over a contour C which is a circle of radius $R \to \infty$ centred at the origin. Here

$$n(z) = \left(\frac{1}{e^{\hbar z/k_B T} - 1}\right),$$

which you might recognize as the Bose distribution function (phonons are bosons) at zero chemical potential with $\hbar z$ playing the role of energy. How is the sum related to the singularities of $f(z)$ and their residues?

(b) Now, consider a specific case of the function f, which is

$$f(i\omega_n) = \frac{2\Omega}{\omega_n^2 + \Omega^2}\frac{1}{ip_m + i\omega_n - E},$$

where Ω and E are real numbers and p_m is also a Matsubara frequency. Evaluate

$$S = -\frac{1}{k_B T}\sum_n f(i\omega_n).$$

Note that the summation is over n and not m, which is some fixed integer. Physically, the parameters Ω and E are the frequency of the phonon and the energy of the electron respectively though that is not important to performing the summation.

(c) Suppose the summation had to be performed over all Matsubara frequencies of the form $\omega_n = (2n+1)\pi k_B T/\hbar$. What would be an appropriate choice of $n(z)$ in part (a)?

5. The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \int_0^\infty f(x) e^{-sx} dx.$$

(a) Show that the Laplace transform of the function

$$f(x) = \frac{e^{ax}}{\sqrt{x^3}},$$

is

$$F(s) = -2\sqrt{\pi}\sqrt{s-a},$$

for $s > a$. You do not need to use contour integration for this: Use known results for Gaussian integrals and manipulate them appropriately.

(b) The inverse Laplace transformation can be shown to be defined in terms of the Bromwich integral

$$f(x) = \frac{1}{2\pi i}\int_{\lambda - i\infty}^{\lambda + i\infty} F(s) e^{sx} ds,$$

where the integral is along a line parallel to the imaginary axis. λ is taken to be any real positive number larger than the real parts of all the singularities of $F(s)e^{sx}$. Evaluate the inverse Laplace transform of $F(s)$ given in part (a) and verify that it is equal to the given function $f(x)$. To do this, you will need to choose an appropriate contour to evaluate the Bromwich integral. Note that the $F(s)$ taken to be a function of a complex variable z has a branch point at $z = a$.