PH 205: Mathematical methods of physics

Problem Set 8

1. The total solid angle subtended by any closed volume about a point in its interiors in three dimensions is 4π. In two dimensions, the analogous statement is that the total angle subtended by a closed surface about an interior point is 2π. It is possible to define a "solid angle' in n dimensions, where n is a positive integer. To do this, consider a d dimensional spherical surface centred at the origin. An infinitesimal element inside this sphere at a distance r from the origin has a volume

$$dV = r^{n-1} dr d\Omega,$$

in "spherical coordinates", where $d\Omega$ is the infinitesimal solid angle subtended by the element. The goal of this problem is to calculate $\int_S d\Omega$ where the integral is over the surface of the sphere and independent of its radius (which can also be infinite).

   (a) Consider the integral

$$\int\int \ldots \int \exp(-\sum_{i=1}^{n} x_i^2) dx_1 dx_2 \ldots dx_n,$$

   where $x_1, x_2 \ldots x_n$ are the cartesian coordinates along the n dimensions and the integral is over all space. Write this integral in spherical coordinates. Evaluate the integral in both cartesian and spherical coordinates. Since the two integrals have to have the same value, show that you obtain the value of $\int_S d\Omega$. Show that the resultant expression involves the gamma function.

   (b) Use the above result to calculate the volume of an n dimensional sphere of radius R.

2. It was mentioned in class that there are applications in physics that require the zeta function $\zeta(-n)$ to have a finite value for non-negative integers n. In this problem you will work out one way to obtain such values. Consider the sum

$$S(\epsilon) = \sum_{k=1}^{\infty} e^{-k\epsilon},$$

where k takes on all positive values and ϵ is a positive number

   (a) How will you relate $\zeta(-m)$ for a non-negative integer m to $S(\epsilon)$ in the limit $\epsilon \to 0$?

   (b) Now, using the above relation calculate $\zeta(0)$ and $\zeta(-1)$. You will see that these sums have divergent terms when $\epsilon \to 0$. Subtraction of these divergent terms gives you the desired answers. In actual physics problems involving the zeta function, there are physical reasons for this subtraction.

   (c) Will the same kind of trick work for $\zeta(n)$ for positive integers n if you find some way to relate $S(\epsilon)$ to $\zeta(n)$? Why (not)?

3. In this problem you will obtain the condensation temperature of bosons in different potentials and dimensions using the zeta function. The Bose distribution function

$$f_B(x) = \frac{1}{e^x - 1},$$

gives the occupancy of a level of energy E by bosons at a temperature T and chemical potential μ when $x = (E - \mu)/k_B T$,. k_B is the Boltzmann constant. If N is the number of bosons present in the system,

$$N = \sum_E f_B[(E - \mu)/k_B T],$$

where the sum is over all possible energy eigenstates of a boson in the given potential. The above equation can be used to fix the value of μ given N, T and the energy eigenvalues. It can be seen from the form of $f_B(x)$ that μ has to always be less than or equal to the lowest value of energy E [$f_B(x)$ being an occupancy cannot be negative]. It can also be shown that for fixed N, μ increases with decreasing temperature. Condensation occurs at a temperature T_c, where μ becomes equal to the lowest value of energy.

(a) First consider the N bosons (of mass m) in a cubic box of side L. Write down an equation for T_c as a sum over the allowed energy states. There is no straightforward way to obtain T_c from the above equation but a simplification can be made in the limit of very large L. Show that in this limit, the sum can be written as an integral. (*Hint: Use the same trick to convert the sum to an integral that was used to convert a Fourier series into a Fourier transform.*)

(b) Obtain T_C in terms of the other parameters from the integral obtained above. Show that this expression involves the zeta function.

(c) Repeat parts (a) and (b) for bosons in a two dimensional square box of side L. What is the value of T_c you get in this case?

(d) In an actual experimental situation, the bosons are placed in a harmonic oscillator potential (and not a box). Assume that for a three dimensional system, the frequency of the harmonic oscillator is ω in all three directions. Once again for N bosons, first write obtain an equation for T_c as a sum over the allowed energy states. This time convert the sum into an integral assuming that $\hbar\omega \ll k_B T$. From the integral obtain an expression for T_C in terms of the other parameters and show that this expression also involves the zeta function.

(e) Repeat (d) for a two dimensional harmonic potential of frequency ω in each direction.

4. Consider a damped one dimensional harmonic oscillator whose equation of motion is given by

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

where the symbols have their usual meanings.

(a) How do you obtain the solutions to the above equation by means of an "educated guess" as you have learnt in your mechanics course? What are the criteria for underdamped, critically damped and overdamped motions?

(b) In each of the three cases above, use the general formula from class to show that each linearly independent solution can be obtained in terms of the other.

5. Consider the Legendre equation which appears in a lot of problems in Physics:

$$\left(1 - x^2\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + l(l+1)y = 0,$$

where l is any real number.

(a) What kind of a point (ordinary, regular singular or irregular) is $x = 0$?

(b) When l is a positive integer, show by expanding about $x = 0$ that one of the solutions is a polynomial of order l, $P_l(x)$ while the other is an infinite series $Q_l(x)$. These polynomials are the Legendre polynomials you have encountered earlier. Calculate $P_0(x)$, $P_1(x)$ and $P_2(x)$ using the series method.

(c) Use the general formula obtained in class to calculate $Q_0(x)$ and $Q_2(x)$ from $P_0(x)$ and $P_1(x)$ respectively.