PH 205: Mathematical methods of physics

Problem Set 9

1. The most general linear second order differential operator $\mathcal{L}$ in two variables x and y has the form of

$$\mathcal{L} = a\frac{\partial^2}{\partial x^2} + 2b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2} + d\frac{\partial}{\partial x} + e\frac{\partial}{\partial y} + f,$$

where a, b, c, d, e and f are real numbers.

   (a) Show that $\mathcal{L}$ can be written in the form

$$\mathcal{L} = \alpha\left(\frac{\partial}{\partial \xi} - \mu\right)^2 + \beta\left(\frac{\partial}{\partial \rho} - \nu\right)^2 + \gamma,$$

   where $\alpha, \beta, \gamma, \mu$ and ν are real numbers that depend on a, b, c, d, e and f and ξ and ρ are variables that are linearly dependent on x and y.

   (b) Argue that the classification of the equation as hyperbolic, parabolic or elliptic depends only on the coefficients a, b and c and not d, e and f.

2. The Helmholtz equation for a function ϕ is the homogeneous partial differential equation

$$\nabla^2\phi + m^2\phi = 0,$$

where m is a real number. Consider the Helmholtz equation inside a cube of side a centred at the origin. The boundary conditions are that $\phi = 0$ on every face of the cube.

   (a) What is $\phi(x, y, z)$ at every point inside the cube for different values of m?

   (b) Is the above solution unique for a given m?

   (c) What is the solution for $m = 0$? Is it unique?

3. Consider a quantum mechanical particle of mass μ in a spherical box of radius a, i.e. the potential is zero inside the box and infinite outside.

   (a) Write the Schrödinger equation for this problem in spherical polar coordinates. Separate the equation out into radial and angular parts.

   (b) The solution of the angular part will give you the spherical harmonics $Y_{lm}(\theta, \phi)$ as expected for any problem with a spherically symmetric potential. Further the energy eigenvalues will be independent of m. Show that the a general wavefunction corresponding to an energy eigenvalue E and with quantum numbers l and m has the form

$$\psi(r, \theta, \phi) = R_l(\sqrt{2mE}r/\hbar)Y_{lm}(\theta, \phi),$$

   where R_l is the solution to the radial equation and depends on the eigenvalue l.

   (c) Show using an appropriate substitution that the radial equation can be transformed into the Bessel equation and hence $R_l(\sqrt{E}r)$ can be written in terms of a Bessel function.

   (d) It can be argued that the ground state of the particle has $l = 0$. Calculate the ground state energy and wavefunction.

4. The equation of motion of a one dimensional forced damped harmonic oscillator is

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t),$$

where the symbols have their usual meanings. Assume that the initial conditions are $x(t = 0) = 0$ and $x'(t = 0) = 0$. The solution to this equation is of the form

$$x(t) = \int_{-\infty}^{\infty} G(t, t')F(t')\ dt',$$

where $G(t, t')$ is the Green's function.

(a) Calculate $G(t, t')$ in two ways: 1) obtaining the solution for $t < t'$ and $t > t'$ and matching them appropriately at $t = t'$ and 2) using Fourier transforms. Do you get the same result both ways?

(b) Calculate $x(t)$ for the cases: 1) $F(t) = F_0\delta(t)$, 2) $F(t) = F_0 \cos \Omega t$ and 3) $F(t) = F_0 e^{-t/\tau}$.

5. For the diffusion equation and the free particle Schrödinger equation both in one dimension, show explicitly that the solution at at a point x and time t can be obtained as an integral over an initial value of the function involving the Green's function.