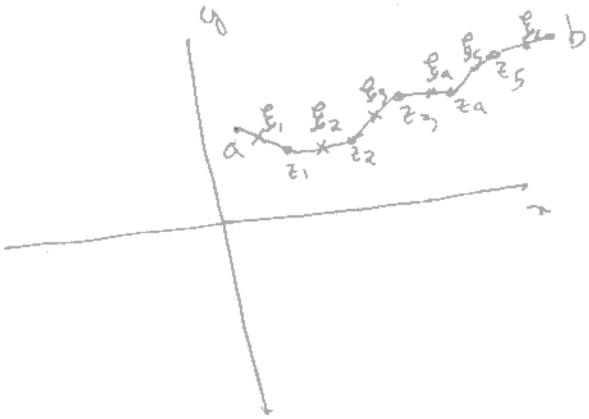
$$I = \int_a^b f(z)\,dz$$



$$I = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) \qquad z_0 = a,\ z_n = b$$

$$= \sum_{k=1}^{n} f(\xi_k)\,\Delta z$$

$\Delta z \to 0$

$$I = \int_a^b f(z)\,dz$$

$f(z) = u(x,y) + i\,v(x,y)$; $dz = dx + i\,dy$

$$I = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$$ C denotes the curve from a to b

Properties

(a) $\displaystyle\int_C \{f(z) + g(z)\}\,dz = \int_C f(z)\,dz + \int g(z)\,dz$

(b) $\displaystyle\int_C A\,F(z)\,dz = A\int_C f(z)\,dz$. A is a constant

(c) $\int_a^b f(z)\,dz = -\int_b^a f(z)\,dz$ along the same curve

(d) $\int_a^b f(z)\,dz = \int_a^m f(z)\,dz + \int_m^b f(z)\,dz$ a, b, m are points on C

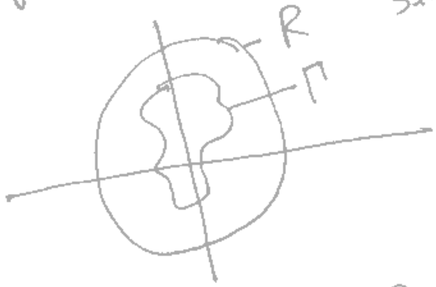
(e) $\left|\int_C f(z)\,dz\right| \leq ML$

if $|f(z)| \leq M$ everywhere on C and $L = \int_C |dz|$

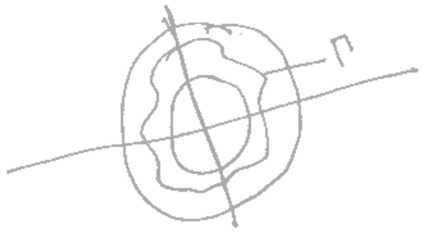
Simply and multiply connected regions

R is a simply connected region if any closed curve which lies in R can be shrunk to a point without leaving R

simply connected



multiply connected



Jordan curve theorem

A continuous closed non-intersecting curve is called a Jordan curve

Theorem: A Jordan curve divides the plane into two regions: (i) A simply connected interior with points z satisfying $|z| < M$, where M is some positive number and an exterior.

Very hard to prove!

Integrals over closed curves

$\oint_C f(z)\,dz$, Convention +ve direction: walk along path, interior region to your left.

Cauchy-Goursat theorem

If $f(z)$ is analytic in a region R and its boundary C.

$$\oint_C f(z)\,dz = 0$$

Morera's theorem (converse of the CG theorem)

If $f(z)$ is continuous in R and

$$\oint_C f(z)\,dz = 0 \quad \text{for every closed curve in R.}$$

$f(z)$ is analytic in R.

Cauchy - Goursat theorem: Cauchy's proof, when $f'(z)$ is also continuous (assumed)

$$f(z) = u(x,y) + i\,v(x,y)$$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \text{(CR conditions)}$$

$$\oint f(z)\,dz = \oint_c (u+iv)(dx+idy)$$

$$= \oint u\,dx - v\,dy + i\oint_C v\,dy + u\,dx$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$

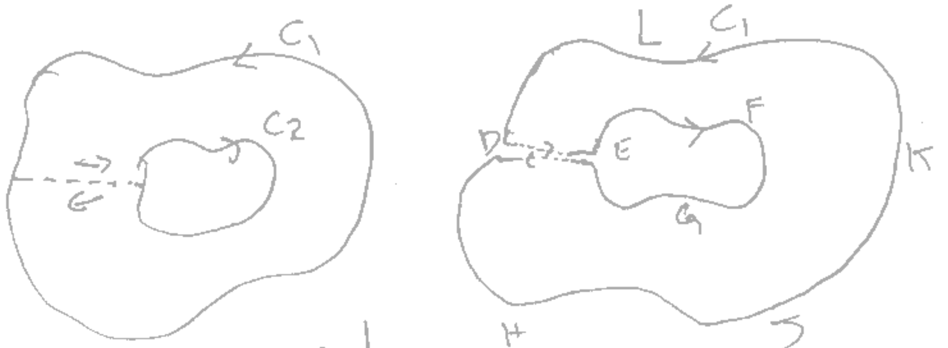
(Green's theorem or stokes theorem applicable when $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$ are continuous)

$\oint f(z)\,dz = 0$ from the CR conditions

Goursat removed the explicit assumption of $f'(z)$, being continuous.

In fact, we know that $f(z)$ ~~cont~~ analytic in $R \Rightarrow$ $f'(z)$ is continuous in R.

$f(z)$ is analytic in ~~an~~ region bounded by two closed curves C_1 and C_2



$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

$$\oint f(z)\,dz = 0 = \int_{\substack{\cancel{DEFGHJKLD}\\ DEFGEDHJKLD}} f(z)\,dz = 0$$

$$= \int_{DE} f(z)\,dz + \int_{EFGE} f(z)\,dz + \int_{ED} f(z)\,dz + \int_{DHJKLD} f(z)\,dz = 0$$

$$\int_{DE} = -\int_{ED} \; ; \; \int_{EFGE} = \int_{C_2} \; ; \; \int_{DHJKLD} = \int_{C_1}$$

$$\Rightarrow \oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz$$

Indefinite integrals

$f(z)$ and $F(z)$ are analytic in R : $F'(z) = f(z)$

$$F(z) = \int f(z)\,dz + C$$

$$\int z^n\,dz = \frac{z^{n+1}}{n+1} \;;\; n = -1$$

$$\int e^z\,dz = e^z$$

$$\int \sin z\,dz = -\cos z$$

$$\int \cos z\,dz = \sin z$$

⋮

Consequence of Cauchy's theorem

$f(z)$ is analytic in a simply connected region R

$\int_{z_1}^{z_2} f(z)\,dz$ is independent of the path

joining z_1 to z_2 in R.

Cauchy's integral formula

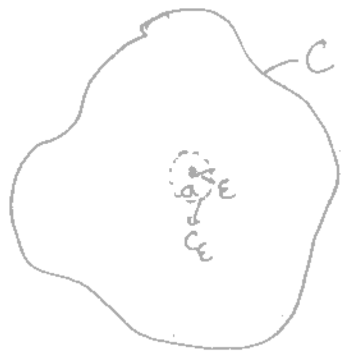
$f(z)$ is analytic ~~on~~ inside ~~and~~ on a Jordan curve R
and a is any point inside C

$$\Rightarrow \quad f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

The n^{th} derivative

$$f^{n}(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz$$

Proof



Encircle a by a circle of radius ε, $\frac{f(z)}{z-a}$ is analytic in the region R', between the circle and R.

$$\Rightarrow \quad \oint_C \frac{f(z)}{z-a}\, dz = \oint_{C_\varepsilon} \frac{f(z)}{z-a}\, dz$$

$$z-a = \varepsilon e^{i\theta} \Rightarrow dz = i\varepsilon e^{i\theta}\, d\theta$$

$$\oint_{C_\varepsilon} \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+\varepsilon e^{i\theta})\, i\varepsilon e^{i\theta}}{\varepsilon e^{i\theta}}\, d\theta = i\int_0^{2\pi} f(a+\varepsilon e^{i\theta})\, d\theta$$

$$\varepsilon \to 0 \Rightarrow \oint \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} \lim_{\varepsilon \to 0} f(a+\varepsilon e^{i\theta})\, d\theta = 2\pi i f(a)$$

$\because$ $f(z)$ is continuous in R.

Thus $f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$

$$f'(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{(z-a)^2}\,dz$$

Proof:

$$f'(a) = \underset{h\to 0}{\frac{f(a+h)-f(a)}{h}}$$

$$f(a+h) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-(a+h)}\,dz \; ; \; f(a) = \frac{1}{2\pi i}\oint \frac{f(z)}{z-a}\,dz$$

$$f'(a) = \lim_{h\to 0}\frac{1}{2\pi i}\oint_C \frac{1}{h}\left\{\frac{1}{z-(a+h)} - \frac{1}{z-a}\right\} f(z)\,dz$$

$$= \lim_{h\to 0}\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-a-h)(z-a)}$$

$$\frac{1}{2\pi i}\oint \frac{f(z)\,dz}{(z-a)^2} + \frac{h}{2\pi i}\oint_C \frac{f(z)}{(z-a-h)(z-a)^2}\,dz$$

$$\oint_C \frac{f(z)}{(z-a-h)(z-a)^2}\,dz = \int_\Gamma \frac{f(z)}{(z-a-h)(z-a)^2}\,dz$$

Γ is a circle of radius ε

$|h| < \frac{\varepsilon}{2}$ ($\because$ $a+h$ also lies inside Γ)

$|z-a-h| \geq |z-a| - |h| > \varepsilon - \varepsilon/2 = \varepsilon/2$

Thus

$$\left| \frac{h}{2\pi i} \oint \frac{f(z)\,dz}{(z-a-h)\,(z-a)^2} \right| \leq \frac{|h|}{2\pi} \frac{M\, 2\pi\varepsilon}{\frac{\varepsilon}{2}\,\varepsilon^2} = \frac{2hM}{\varepsilon^2} \to 0$$

as $h \to 0$

$|f(z)| \leq M$ inside Γ for some M, which is true since

$f(z)$ is analytic inside M.

$$\frac{d}{da} f(a) = \frac{d}{d}\left\{ \frac{1}{2\pi i} \oint \frac{f(z)}{z-a}\,dz \right\} = \frac{1}{2\pi i} \oint_C \frac{\partial}{\partial a}\left\{ \frac{f(z)}{z-a} \right\} dz$$

$$= \frac{1}{2\pi i} \int \frac{f(z)}{(z-a)^2}\,dz \quad \text{Leibnitz's theorem}$$

Using induction

$$f^n(a) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Liouville's theorem: If for all z in the entire complex plane $f(z)$ is analytic and $|f(z)| < M$

$\Rightarrow$ $f(z)$ is a constant

Fundamental theorem of algebra

Every polynomial of the form

$$p(z) = \sum_{k=0}^{n} a_k z^k \quad \text{where } n \geq 1 \text{ and } a_n \neq 0$$

has at least one root.

Proof: Consider $f(z) = \frac{1}{p(z)}$. Assume $p(z)$ has no root

Thus $f(z)$ is analytic $\forall z$. Also $|f(z)| = \frac{1}{|p(z)|}$

is bounded $\Longrightarrow$ $f(z)$ and thus $p(z)$ is a constant

from Liouville's theorem. Contradiction!!

Thus $p(z)$ has at least one root.

Corollary: $p(z)$ has exactly n roots.

Let α be a root of $p(z)$

$p(z) - p(\alpha) = (z-\alpha)\, q(z)$ where $q(z)$ is a polynomial

of degree $n-1$. Thus it has at least one root.

Keep going till n roots are obtained

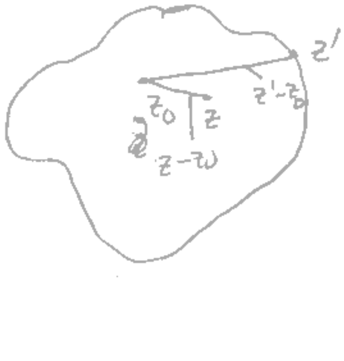
Taylor series

Consider a function $f(z)$ that is analytic in a

region bounded by a curve C.

Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{z'-z}$$



$$= \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z'-z_0)-(z-z_0)}$$

$$= \frac{1}{2\pi i} \oint_C \frac{f(z')}{z'-z_0} \frac{1}{\left(1 - \frac{z-z_0}{z'-z_0}\right)} dz'$$

$$|z-z_0| < |z'-z_0|$$

Thus $$\frac{1}{1-\frac{z-z_0}{z'-z_0}} = 1 + \frac{z-z_0}{z'-z_0} + \left(\frac{z-z_0}{z'-z_0}\right)^2 + \cdots\cdots$$

$$f(z) = \frac{1}{2\pi i}\oint_C \sum_{n=0}^{\infty} \frac{f(z')}{z'-z_0}\left(\frac{z-z_0}{z'-z_0}\right)^n dz'$$

$$= \frac{1}{2\pi i}\sum_{n=0}^{\infty}(z-z_0)^n \oint \frac{f(z')\,dz'}{(z'-z_0)^{n+1}}$$ [$\oint_C$ and $\sum_n$ interchange allowed due to uniform convergence]

$$= \sum_{n=0}^{\infty} \frac{f^n(z_0)}{n!}(z-z_0)^n$$

Taylor expansion

Radius of convergence; Maximum distance from z_0 up to which $f(z)$ remains analytic

<u>Laurent series</u>

Expansion in an annular region between two curves C_1 & C_2.
$f(z)$ analytic in that region

$$f(z) = \frac{1}{2\pi i}\oint_{C_1} \frac{f(z')}{z'-z}dz' - \frac{1}{2\pi i}\oint_{C_2}\frac{f(z')\,dz'}{z'-z}$$

$$z' - z = (z' - z_0) - (z - z_0)$$

$|z' - z_0| < |z - z_0|$ for the integral over C_2

$|z' - z_0| > |z - z_0|$ for the integral over C_1

$$f(z) = \frac{1}{2\pi i} \sum_{n=0}^{\infty} (z - z_0)^n \oint_{C_1} \frac{f(z')\,dz'}{(z' - z_0)^{n+1}} + \frac{1}{2\pi i} \sum_{n=1}^{\infty} (z - z_0)^{-n} \oint_{C_2} (z' - z_0)^{n-1} f(z')\,dz'$$

Laurent series

Taylor series

If $f(z)$ is analytic inside C_2

$$\oint (z' - z_0)^{n-1} f(z')\,dz = 0 \quad \forall\, n \geq 1$$

and only the Taylor series remains

$$f(z) = \sum_{n=-\infty}^{\infty} (z - z_0)^n a_n$$

$$a_n = \frac{1}{2\pi i} \oint_C \frac{f(z')\,dz'}{(z' - z_0)^{n+1}}$$

Choice of C does not matter. Any C encircling z_0 inside the annular region will do.

Example

$\dfrac{1}{z(z-1)}$ about $z=0$ in the annular region between circles of radius 0 and 1

$$\frac{1}{z(z-1)} = \frac{1}{1-z} - \frac{1}{z} = 1+z+z^2+z^3+\cdots - \frac{1}{z}$$

can also be obtained from the integral formula

If one expands $f(z)$ about an isolated singularity at z_0 where it has a pole of order n, the Laurent expansion stops at $\dfrac{A}{(z-z_0)^n}$

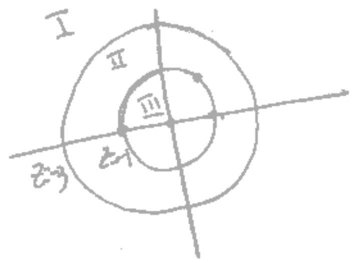
Essential singularity: The Laurent expansion does not stop

e.g. $f(z) = e^{1/z}$ about $z=0$

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots\cdots$$

Laurent series ~~eg.~~

$$f(z) = \frac{1}{(z+1)(z+3)}$$



~~Laurent series in region~~

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z+3}\right]$$

Region I

$$|z| > 3$$

$$\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots\cdots$$

$$\frac{1}{z+3} = \frac{1}{z\left(1+\frac{3}{z}\right)} = \frac{1}{z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right]$$

$$f(z) = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \frac{40}{z^5}$$

Region II

$$1 < |z| < 3$$

$$\frac{1}{z+1} = \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \frac{1}{z^4} + \cdots$$

$$\frac{1}{z+3} = \frac{1}{3}\,\frac{1}{1+\frac{z}{3}} = \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$\cancel{f(z) = -\frac{1}{2z^4} + 1}$$

$$f(z) = \cdots\cdots - \frac{1}{2z^4} + \frac{1}{2z^3} - \frac{1}{2z^2} + \frac{1}{2z} - \frac{1}{6} + \frac{z}{18} - \frac{z^2}{54} + \frac{z^3}{162} - \cdots$$

Region III $|z| < 1$

$$\frac{1}{z+1} = 1 - z + z^2 - z^3 + \cdots$$

$$\frac{1}{z+3} = \frac{1}{3}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \frac{40}{81}z^3 + \cdots$$

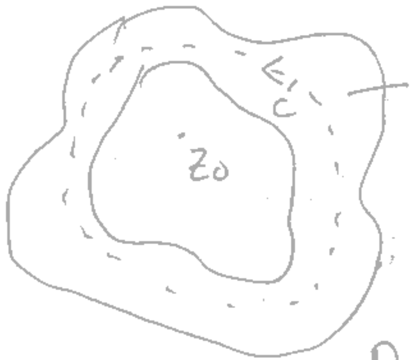
Taylor series since $f(z)$ is analytic in region I.

Residue theorem

Let the Laurent expansion of a function ~~in some region containing~~

be $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ in some region

$$\oint_C f(z)\,dz = \sum_{n=-\infty}^{\infty} a_n \oint_C (z-z_0)^n\,dz \quad \text{where } C \text{ ~~contains~~ encircles } z_0$$



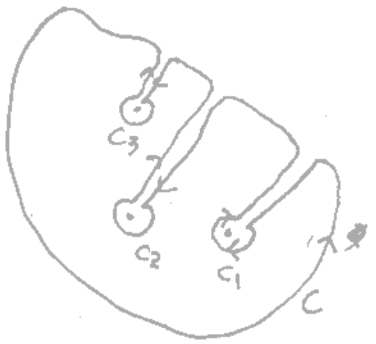
Region where Laurent expansion is defined

$$\oint f(z)\,dz = 2\pi i\, a_{-1} \quad \text{— Residue theorem}$$

$$\oint_C (z-z_0)^n\,dz = 0 \quad \forall\, n \neq 1$$
$$= 2\pi i \quad \text{for } n = 1$$

a_1 is called the residue.

Isolated singularities



$$\oint_C f(z)\,dz + \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz + \oint_{C_3} f(z)\,dz + \cdots = 0$$

$$\oint_{C_j} f(z)\,dz = -2\pi i\, a_{-1,j}$$

$$\Rightarrow \oint f(z)\,dz = 2\pi i \sum_j a_{-1,j} = 2\pi i \text{ (Sum of enclosed residues.)}$$

Evaluation of residues

$$f(z) = \cdots\, a_n (z-z_0)^{-n} + \cdots a_{-2}(z-z_0)^{-2} + a_{-1}(z-z_0)^{-1}$$
$$+ a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots\, a_n (z-z_0)^n + \cdots$$

Pole of order 1 $\quad a_{-1} = \lim_{z\to z_0} (z-z_0) f(z)$

Pole of order n ??

$$a_{-1} = \frac{1}{(n-1)!} \lim_{z\to z_0} \left[\frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z) \right] \right]$$

Example

$$f(z) = \frac{z}{(z-1)^2} \qquad \text{Residue at } z=1\,?$$

Let $z-1=\omega$

$$f(z) = \frac{\omega+1}{\omega^2} = \frac{1}{\omega} + \frac{1}{\omega^2} = \frac{1}{z-1} + \frac{1}{(z-1)^2}$$

Residue at $z=1$ is 1

Using the formula

$$a_1 = \frac{1}{(n-1)!} \lim_{z\to z_0} \frac{d^{n-1}}{dz^{n-1}} \left[(z-z_0)^n f(z)\right]$$

$n=2$ for this pole

$$a_{-1} = \lim_{z\to 1} \frac{dz}{dz} = 1$$

eg $f(z) = \dfrac{\cot \pi z}{z(z+2)}$ at $z=0$ $\qquad f(z) = \dfrac{1}{z \tan \pi z\,(z+2)}$

Pole of order 2 at $z=0$

Application of the formula for a_{-1} is involved.

Simpler method:

Around $z=0$

~~$\tan \pi z \approx \pi z + 0$~~

~~$z+2 = 2\left(1+\frac{z}{2}\right) =$~~

$$\cot(\pi z) = (\pi z)^{-1} + O(z)$$

$$\frac{1}{z+2} = \frac{1}{2}\left(1+\frac{z}{2}\right)^{-1} = \frac{1}{2}\left[1-\frac{z}{2}+\frac{z^2}{4}+\cdots\right]$$

$$= \frac{1}{2}\left[1-\frac{z}{2}+O(z^2)\right]$$

$$f(z) = \frac{1}{z}\left[(\pi z)^{-1} + O(z)\right] \frac{1}{2}\left[1-\frac{z}{2}+O(z^2)\right]$$

$$= \frac{1}{2\pi z^2} - \frac{1}{4\pi z} + O(1)$$

Residue $= -\frac{1}{4\pi}$

Cauchy principal value.

Isolated pole directly in path of integration

Consider

$$\int_{-a}^{b} \frac{dx}{x} \neq \ln b - \ln a$$

$\because$ 0 is in the path of integration and $\ln x$ blows up at zero.

$-a \quad 0 \quad b$

$$\lim_{\delta_1, \delta_2 \to 0} \int_{-a}^{-\delta_1} \frac{dx}{x} + \int_{\delta_2}^{b} \frac{dx}{x} = \ln b - \ln \delta_2 + \ln \delta_1 - \ln a$$

in general divergent

However if $\delta_1 = \delta_2$ $\quad \int_{-a}^{b} \frac{dx}{x} = \ln b - \ln a$.

$$\int_{-a}^{b} \frac{dx}{x} \equiv \lim_{\delta \to 0} \int_{-a}^{-\delta} \frac{dx}{x} + \int_{\delta}^{b} \frac{dx}{x} = \ln b - \ln a = P\int_{-a}^{b} \frac{dx}{x}$$

P – Cauchy principal value

Example $\quad \displaystyle\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \int_{0}^{\infty} \frac{e^{ix} - e^{-ix}}{2ix}\, dx$

$$\stackrel{?}{=} \int_{0}^{\infty} \frac{e^{ix}}{2ix}\, dx + \int_{0}^{\infty} \frac{e^{-ix}}{2ix}\, dx$$

Each integral is logarithmically divergent at $x=0$

Define $I_2 = \int_0^\infty \frac{e^{-ix}}{2ix}\,dx$ as $\lim_{\delta\to 0}\int_\delta^\infty \frac{e^{-ix}}{2ix}\,dx$

$x \to -ix \Rightarrow I_2 = \int_{-\infty}^{-\delta} \frac{e^{ix}}{2ix}\,dx$

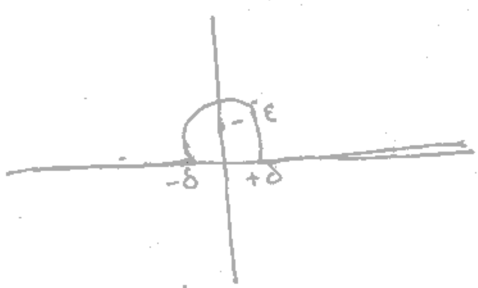
|||ly define $I_1 = \int_0^\infty \frac{e^{ix}}{2ix}\,dx$ as $\lim_{\delta\to\infty}\int_\delta^\infty \frac{e^{ix}}{2ix}\,dx$

Thus $\int_0^\infty \frac{\sin x}{x}\,dx = \lim_{\delta\to 0}\left[\int_{-\infty}^{-\delta}\frac{e^{ix}}{2ix}\,dx + \int_\delta^\infty \frac{e^{ix}}{2ix}\,dx\right]$

$$= P\int_{-\infty}^{\infty}\frac{e^{ix}}{2ix}\,dx$$

Important identity

$$\lim_{\varepsilon\to 0}\frac{1}{x-i\varepsilon} = P\left(\frac{1}{x}\right) + i\pi\delta(x)$$

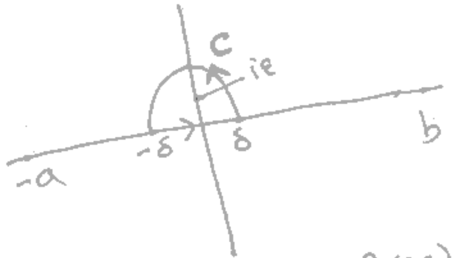


Consider $I = \int_{-a}^{b}\frac{f(x)}{x-i\varepsilon}\,dx$. There is now no pole along the path of integration

Encircle the pole with a half circle of radius ($\delta>\varepsilon$) as shown

$$\int_{-a}^{b} \frac{f(x)}{x-i\epsilon}\,dx = \int_{-a}^{-\delta}\frac{f(x)}{x-i\epsilon}\,dx + \int_{-\delta}^{\delta}\frac{f(x)}{x-i\epsilon}\,dx + \int_{\delta}^{b}\frac{f(x)}{x-i\epsilon}\,dx$$

Consider $\displaystyle\int_{-\delta}^{\delta}\frac{f(x)\,dx}{x-i\epsilon}$



$$\oint_C \frac{f(z)}{z-i\epsilon}\,dz = 2\pi i\, f(i\epsilon) \quad \text{if } f(i\epsilon) \text{ is } \cancel{\text{analytic}} \text{ analytic}$$

on and inside C.

Suppose $f(z)$ is ~~smooth~~ at $z=0$.

$$\lim_{\epsilon\to 0}\oint_C \frac{f(z)}{z-i\epsilon}\,dz = 2\pi i f(i\epsilon) = 2\pi i\, f(0)$$

$$\oint_C \frac{f(z)}{z-i\epsilon}\,dz = \int_{-\delta}^{\delta}\frac{f(x)}{x-i\epsilon}\,dx + \int_{\theta=0}^{\pi}\frac{f(\delta e^{i\theta})}{\delta e^{i\theta}-\epsilon}\,\delta i e^{i\theta}\,d\theta$$

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{0}^{\pi}\frac{f(\delta e^{i\theta})}{\delta e^{i\theta}-\epsilon}\,\delta i e^{i\theta}\,d\theta = i\pi f(0)$$

$$\lim_{\epsilon\to 0}\oint_C \frac{f(z)}{z-i\epsilon}\,dz = 2\pi i\, f(0)$$

$$\Rightarrow \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-\delta}^{\delta}\frac{f(x)}{x-i\epsilon}\,dx = i\pi f(0)$$

$$\lim_{\delta\to 0}\lim_{\epsilon\to 0}\left[\int_{-a}^{-\delta}\frac{f(x)}{x-i\epsilon}\,dx + \int_{\delta}^{b}\frac{f(x)}{x-i\epsilon}\,dx\right] = P\int_{-a}^{b}\frac{f(x)}{x}\,dx$$

Thus

$$\lim_{\varepsilon \to 0} \int_{-a}^{b} \frac{f(x)}{x - i\varepsilon}\, dx = P\int_{-a}^{b} \frac{f(x)}{x}\, dx + i\pi f(0)$$

or $$\lim_{\varepsilon \to 0} \frac{1}{x - i\varepsilon} = P\left(\frac{1}{x}\right) + i\pi\, \delta(x).$$

Generally, $\lim_{\varepsilon \to 0}$ is not written explicitly in the above formula

or $$\frac{1}{x - i\varepsilon} = P\left(\frac{1}{x}\right) + i\pi\delta(x)$$

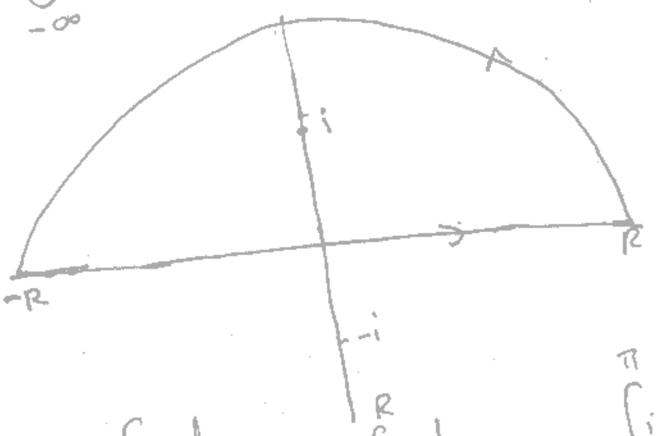
$$\text{|||ly} \quad \frac{1}{x + i\varepsilon} = P\left(\frac{1}{x}\right) - i\pi\delta(x)$$

$$\frac{1}{x - i\varepsilon} = \frac{x + i\varepsilon}{x^2 + \varepsilon^2}$$

Recall $\delta(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}$

Evaluation of definite integrals

Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}$



Consider $$\oint_C \frac{dz}{1 + z^2} = \int_{-R}^{R} \frac{dx}{1 + x^2} + \int_{\theta=0}^{\pi} \frac{iRe^{i\theta}\, d\theta}{1 + R^2 e^{i2\theta}} = 2\pi i \text{ Residue at } z = +i$$

$$= 2\pi i \left(\frac{-i}{2}\right)$$

$$= \pi$$

$$\lim_{R\to\infty}\oint_C \frac{dz}{1+z^2} = \lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{1+x^2} + \lim_{R\to\infty} iR\int_{\theta=0}^{\pi}\frac{e^{i\theta}\,d\theta}{1+R^2e^{i2\theta}} = \pi$$

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$$

$$\frac{1}{R}\int_{\theta=0}^{\pi} e^{-i\theta}\,d\theta \longrightarrow 0$$

Thus $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \pi$

$x = \tan\theta$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{1+x^2} = -\int_{\pi/2}^{\pi/2}\frac{\sec^2\theta\,d\theta}{\sec^2\theta} = \pi$$

Same result if contour had been chosen to be the lower half circle.

Evaluate $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^4}$

Consider $\displaystyle\oint_C \frac{dz}{1+z^4}$

$\dfrac{1}{1+z^4}$ has four poles of order 1 at $\dfrac{\pm 1 \pm i}{\sqrt{2}}$

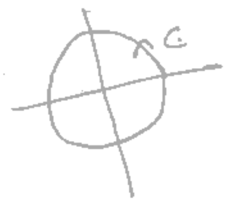
$$\int_{-\infty}^{\infty}\frac{dx}{1+x^4} = 2\pi i \sum \text{residues at } \frac{-1+i}{\sqrt{2}} \;\&\; \frac{1+i}{\sqrt{2}}$$

Trigonometric integrals of range 0 to 2π

$$I = \int_0^{2\pi} \frac{d\theta}{1+a\cos\theta} \qquad |a|<1$$

$\cos\theta = \dfrac{z+z^{-1}}{2}$ if z lies on the unit circle $z=e^{i\theta}$

$$dz = ie^{i\theta}d\theta = iz\,d\theta$$



$$\text{Thus } I = \oint_C \frac{dz}{iz\left[1+a\frac{z+z^{-1}}{2}\right]}$$

$$= -\frac{i2}{a}\oint \frac{dz}{z^2+(2/a)z+1}$$

Simple poles $z_1 = \dfrac{-1+\sqrt{1-a^2}}{a}$ & $z_2 = -\dfrac{1-\sqrt{1-a^2}}{a}$

z_2 is within the unit circle and z_2 outside

$$I = -i\frac{2}{a}\cdot 2\pi i\,\frac{1}{z_2-z_1} = \frac{2\pi}{\sqrt{1-a^2}};\ a<1$$

Evaluation of integrals of the type

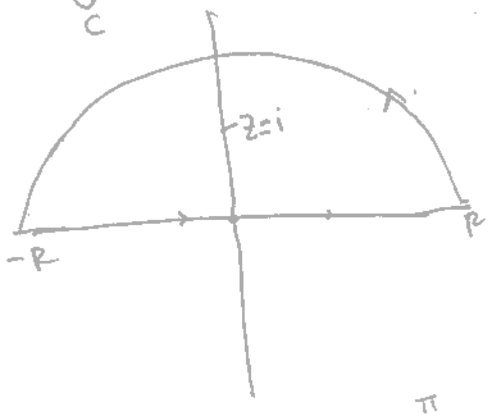
$$I = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \quad \text{– Fourier transforms}$$

Fourier transform of the Lorentzian

$$I = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+1}\, dx$$

Consider

$$I = \oint_C \frac{e^{-ikz}}{z^2+1}\, dz$$



$$I = \int_{-R}^{R} \frac{e^{-ikx}}{x^2+1}\, dx + iR \int_{\theta=0}^{\pi} \frac{e^{-ikRe^{i\theta}}}{R^2 e^{i2\theta}+1} e^{i\theta}\, d\theta$$

Consider $I_R = \lim_{R\to\infty} iR \int_{\theta=0}^{\pi} \frac{e^{-ik(R\cos\theta + iR\sin\theta)}}{R^2 e^{i2\theta}+1} e^{i\theta}\, d\theta$

$$\to \frac{i}{R} \int_{\theta=0}^{\pi} e^{-i\theta}\, e^{kR\sin\theta}\, e^{-ikR\cos\theta}\, d\theta$$

$\to 0$ only if $k \leq 0$ $\because \sin\theta > 0$ in upper half plane

Thus contour in upper half plane for $k \leq 0$ and lower half plane for $k \geq 0$

Choosing appropriate contour (say upper half plane for $k \leq 0$)

$$I = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+1}\,dx = 2\pi i \text{ Residue at } z = i$$

$$\frac{e^{ikz}}{z^2+1} = \frac{1}{2i}\, e^{-ikz}\left[\frac{1}{z-i} - \frac{1}{z+i}\right]$$

~~Reduction~~

$$I = \pi e^{k}$$

|||ly for $k \geq 0$

$$I = \pi e^{-k} \quad \text{(choosing contour along lower half plane)}$$

$$\text{Thus } I = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+1}\,dx = \pi e^{-|k|}$$

Can easily be generalized to

$$I = \int_{-\infty}^{\infty} \frac{e^{-ikx}}{x^2+m^2}\,dx = \frac{\pi}{|m|}\, e^{-|mk|}$$

m is real.

~~Fourier transform of the~~

Laplacian operator

∇^2

Green's function defined as

$\nabla^2 G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$

~~Consider~~
Take Fourier transforms of both sides

$$\tilde{G}(\vec{k}) = \frac{1}{(\sqrt{2\pi})^3}\int G(\vec{r}-\vec{r}')\, e^{-i\vec{k}\cdot(\vec{r}-\vec{r}')}\, d^3\vec{r} \quad \text{in 3D}$$

$$\Rightarrow -k^2 G(\vec{k}) = 1 \qquad \because \text{ FT of } \delta(\vec{r}-\vec{r}') = 1$$

$$G(\vec{k}) = -\frac{1}{k^2 (2\pi)^{3/2}}$$

$$\Rightarrow G(\vec{r}-\vec{r}') = -\frac{1}{(\sqrt{2\pi})^3}\int \frac{1}{k^2}\, e^{i\vec{k}\cdot(\vec{r}-\vec{r}')}\, d^3\vec{k}$$

Use spherical polar co-ordinates for $\vec{k}$

$d^3\vec{k} = k^2\, dk \sin\theta\, d\theta\, d\phi$

Take $\hat{z}$ axis along $\vec{r}-\vec{r}'$ without any loss of generality

$$G(\vec{r}-\vec{r}') = -\frac{1}{(\sqrt{2\pi})^3}\left\{\int_{k=0}^{\infty}\left[\frac{1}{k^2}\int_{x=-1}^{1} e^{ikx|\vec{r}-\vec{r}'|}\, dx\right] k^2\, dk\right\}\int_{\phi=0}^{2\pi} d\phi$$

$x = \cos\theta$

$\sin\theta\, d\theta = dx$

$\vec{k}\cdot(\vec{r}-\vec{r}') = kx|\vec{r}-\vec{r}'|$

$$= -\frac{1}{4\pi^2}\int_{k=0}^{\infty} \frac{e^{ik|\vec{r}-\vec{r}'|} - e^{-ik|\vec{r}-\vec{r}'|}}{ik|\vec{r}-\vec{r}'|}\, dk$$

$$G(\vec{r}-\vec{r}') = \frac{-2\,[\text{illegible}]}{4\pi^2|\vec{r}-\vec{r}'|}\int_{k=0}^{\infty}\frac{\sin k|\vec{r}-\vec{r}'|}{k}\,dk$$

Evaluate.

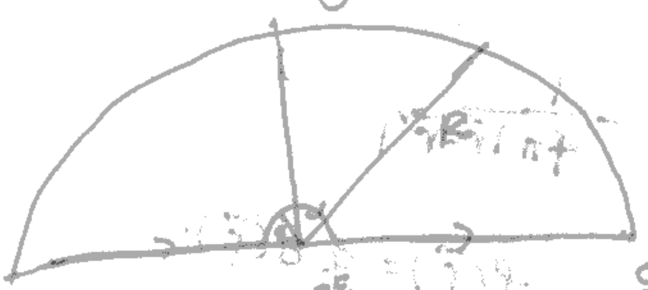
$$I = \int_0^{\infty}\frac{\sin ax}{x}\,dx = \int_0^{\infty}\frac{\sin x}{x}\,dx$$

Substitute $y = ax$

$$I = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$$

Consider $\displaystyle\int_{-\infty}^{\infty}\frac{e^{ix}}{x}\,dx$

Evaluate $\displaystyle\oint\frac{e^{iz}}{z}\,dz$



$$\oint_C\frac{e^{iz}}{z}\,dz = \Bigg[\int_{-R}^{-\varepsilon}\frac{e^{ix}}{x}\,dx + \int_{\theta=\pi}^{0}\frac{e^{i\varepsilon e^{i\theta}}\,i\varepsilon e^{i\theta}\,d\theta}{\varepsilon e^{i\theta}\,d\theta} + \int_{\varepsilon}^{R}\frac{e^{ix}}{x}\,dx + \int_{\theta=0}^{\pi}\frac{e^{iRe^{i\theta}}\,iRe^{i\theta}\,d\theta}{Re^{i\theta}\,d\theta}\Bigg]_{\substack{R\to\infty\\ \varepsilon\to 0}}$$

$$\oint_C \frac{e^{iz}}{z}\,dz = \lim_{R\to\infty} P\int_{-R}^{R} \frac{e^{ix}}{x}\,dx - i\pi$$

$$\searrow P\int_{-\infty}^{\infty}$$

$$\left(\int_{\theta=0}^{\pi} e^{iRe^{i\theta}}\,d\theta \longrightarrow 0 \right)$$

$$\oint \frac{e^{iz}}{z}\,dz = 0 \Rightarrow P\int_{-\infty}^{\infty} \frac{e^{ix}}{x}\,dx = i\pi$$

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \pi \quad \int_{-\infty}^{\infty} \frac{\cos x}{x}\,dx = 0$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

$$G(\vec{r}-\vec{r}\,') = \cancel{\frac{\ldots}{|\vec{r}-\vec{r}\,'|}} = -\frac{1}{4\pi|\vec{r}-\vec{r}\,'|}$$

$$\nabla^2 \varphi(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}, \quad \rho(\vec{r}) = q\,\delta(\vec{r})$$

$$\Rightarrow \varphi(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}$$ – Coulomb's law

Other Green's functions (propagators)

Classical wave equation

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}G(t-t';\vec{r}-\vec{r}') - \nabla^2 G(t-t';\vec{r}-\vec{r}') = \delta(t-t')\delta(\vec{r}-\vec{r}')$$

Diffusion equation

$$\frac{\partial G(t-t';\vec{r}-\vec{r}')}{\partial t} + D\nabla^2 G(t-t';\vec{r}-\vec{r}') = \delta(t-t')\delta(\vec{r}-\vec{r}')$$

Schrödinger equation (free particle)

$$i\hbar\frac{\partial}{\partial t}G(t-t';\vec{r}-\vec{r}') + \frac{\hbar^2}{2m}\nabla^2 G(t-t';\vec{r}-\vec{r}') = \delta(t-t')\delta(\vec{r}-\vec{r}')$$

Helmholtz equation

$$(\nabla^2 + m^2)G(\vec{r},\vec{r}') = \delta(\vec{r}-\vec{r}')$$

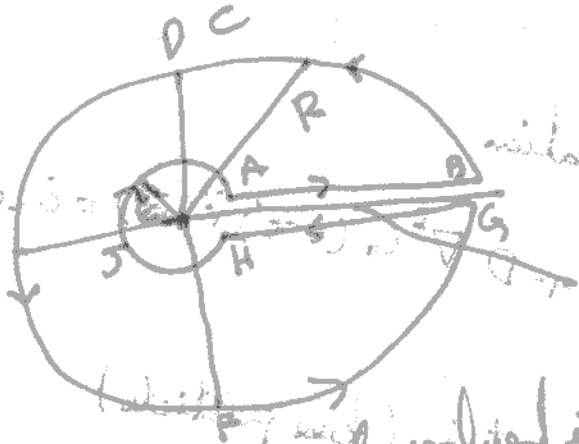
Klein-Gordon equation (free particle)

$$\hbar^2\frac{\partial^2}{\partial t^2}G(t-t';\vec{r}-\vec{r}') - \nabla^2 G(t-t';\vec{r}-\vec{r}') + m^2 G(t-t';\vec{r}-\vec{r}')$$
$$= \delta(t-t')\delta(\vec{r}-\vec{r}')$$

Contour integrals involving branch cuts

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,dx \quad (0 < p < 1)$$

Consider $\oint_C \frac{z^{p-1}}{1+z}\,dz$



Function single valued in this region inside the contour

$$\oint_C = \int_A^B + \int_{BDEFG} + \int_G^H + \int_{HJA}$$

Along AB $z = x$

Along GH $z = x e^{i2\pi}$

$$\int_A^B = \int_A^B \frac{x^{p-1}}{1+x}\,dx$$

$$\int_G^H = -e^{i2\pi(p-1)}\int_A^B \frac{x^{p-1}}{1+x}\,dx \xrightarrow{R\to\infty} -e^{i2\pi(p-1)}\int_0^\infty \frac{x^{p-1}}{1+x}\,dx$$

$$\int_{BDEFG} = \int_0^{2\pi} \frac{R^{p-1} e^{i\theta(p-1)}}{1+Re^{i\theta}} iRe^{i\theta}\, d\theta \xrightarrow[R\to\infty]{} 0 \quad \text{as } R^{(p-1)}$$

$$\int_{HJA} = i\int_{2\pi}^{0} \frac{(\varepsilon e^{i\theta})^{p-1}\, \varepsilon e^{i\theta}\, d\theta}{1+\varepsilon e^{i\theta}} \xrightarrow[\varepsilon\to 0]{} 0$$

$$\oint_C \frac{z^{p-1}}{1+z}\, dz = \left[1-e^{2\pi i(p-1)}\right] \int_0^\infty \frac{x^{p-1}}{1+x}\, dx$$

$= 2\pi i$ Residue at $z=-1$

(Residue at $z=-1$) $= e^{i\pi(p-1)}$

$$\Rightarrow \int_0^\infty \frac{x^{p-1}\, dx}{1+x} = \frac{2\pi i\, e^{i\pi(p-1)}}{1-e^{2\pi i(p-1)}} = \frac{2\pi i}{e^{i\pi p}-e^{-i\pi p}} = \frac{\pi}{\sin p\pi}$$