The solution to this equation is just a harmonic oscillation with a frequency $\omega$, where

$$\omega = \frac{\sqrt{3}l}{m\rho^2} \sin \alpha \tag{8.101}$$

Thus, the circular orbit is stable.

## PROBLEMS

**8-1.** In section 8.2, we showed that the motion of two bodies interacting only with each other by central forces could be reduced to an equivalent one-body problem. Show by explicit calculation that such a reduction is also possible for bodies moving in an external uniform gravitational field.

**8-2.** Perform the integration of Equation 8.38 to obtain Equation 8.39.

**8-3.** A particle moves in a circular orbit in a force field given by

$$F(r) = -k/r^2$$

Show that, if $k$ suddenly decreases to half its original value, the particle's orbit becomes parabolic.

**8-4.** Perform an explicit calculation of the time average (i.e., the average over one complete period) of the potential energy for a particle moving in an elliptical orbit in a central inverse-square-law force field. Express the result in terms of the force constant of the field and the semimajor axis of the ellipse. Perform a similar calculation for the kinetic energy. Compare the results and thereby verify the virial theorem for this case.

**8-5.** Two particles moving under the influence of their mutual gravitational force describe circular orbits about one another with a period $\tau$. If they are suddenly stopped in their orbits and allowed to gravitate toward each other, show that they will collide after a time $\tau/4\sqrt{2}$.

**8-6.** Two gravitating masses $m_1$ and $m_2 (m_1 + m_2 = M)$ are separated by a distance $r_0$ and released from rest. Show that when the separation is $r(< r_0)$, the speeds are

$$v_1 = m_2\sqrt{\frac{2G}{M}\left(\frac{1}{r} - \frac{1}{r_0}\right)}, \qquad v_2 = m_1\sqrt{\frac{2G}{M}\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

**8-7.** Show that the areal velocity is constant for a particle moving under the influence of an attractive force given by $F(r) = -kr$. Calculate the time averages of the kinetic and potential energies and compare with the results of the virial theorem.

**8-8.** Investigate the motion of a particle *repelled* by a force center according to the law $F(r) = kr$. Show that the orbit can only be hyperbolic.

**8-9.** A communications satellite is in a circular orbit around Earth at radius $R$ and velocity $v$. A rocket accidentally fires quite suddenly, giving the rocket an outward radial velocity $v$ in addition to its original velocity.
**(a)** Calculate the ratio of the new energy and angular momentum to the old.
**(b)** Describe the subsequent motion of the satellite and plot $T(r)$, $V(r)$, $U(r)$, and $E(r)$ after the rocket fires.

**8-10.** Assume Earth's orbit to be circular and that the Sun's mass suddenly decreases by half. What orbit does Earth then have? Will Earth escape the solar system?

**8-11.** A particle moves under the influence of a central force given by $F(r) = -k/r^n$. If the particle's orbit is circular and passes through the force center, show that $n = 5$.

**8-12.** Consider a comet moving in a parabolic orbit in the plane of Earth's orbit. If the distance of closest approach of the comet to the Sun is $\beta r_E$, where $r_E$ is the radius of Earth's (assumed) circular orbit and where $\beta < 1$, show that the time the comet spends within the orbit of Earth is given by

$$\sqrt{2(1-\beta)} \cdot (1+2\beta)/3\pi \times 1 \text{ year}$$

If the comet approaches the Sun to the distance of the perihelion of Mercury, how many days is it within Earth's orbit?

8-13. Discuss the motion of a particle in a central inverse-square-law force field for a superimposed force whose magnitude is inversely proportional to the cube of the distance from the particle to the force center; that is,

$$F(r) = -\frac{k}{r^2} - \frac{\lambda}{r^3} \qquad k, \lambda > 0$$

Show that the motion is described by a precessing ellipse. Consider the cases $\lambda < l^2/\mu$, $\lambda = l^2/\mu$, and $\lambda > l^2/\mu$.

**8-14.** Find the force law for a central-force field that allows a particle to move in a spiral orbit given by $r = k\theta^2$, where $k$ is a constant.

**8-15.** A particle of unit mass moves from infinity along a straight line that, if continued, would allow it to pass a distance $b\sqrt{2}$ from a point $P$. If the particle is attracted toward $P$ with a force varying as $k/r^5$, and if the angular momentum about the point $P$ is $\sqrt{k}/b$, show that the trajectory is given by

$$r = b \coth(\theta/\sqrt{2})$$

**8.16.** A particle executes elliptical (but almost circular) motion about a force center. At some point in the orbit a *tangential* impulse is applied to the particle, changing the velocity from $v$ to $v + \delta v$. Show that the resulting relative change in the major and minor axes of the orbit is twice the relative change in the velocity and that the axes are *increased* if $\delta v > 0$.

**8-17.** A particle moves in an elliptical orbit in an inverse-square-law central-force field. If the ratio of the maximum angular velocity to the minimum angular velocity of the particle in its orbit is $n$, then show that the eccentricity of the orbit is

$$\varepsilon = \frac{\sqrt{n} - 1}{\sqrt{n} + 1}$$

**8-18.** Use Kepler's results (i.e., his first and second laws) to show that the gravitational force must be central and that the radial dependence must be $1/r^2$. Thus, perform an inductive derivation of the gravitational force law.

**8-19.** Calculate the missing entries denoted by $c$ in Table 8-1.

**8-20.** For a particle moving in an elliptical orbit with semimajor axis $a$ and eccentricity $\varepsilon$, show that

$$\langle (a/r)^4 \cos\theta \rangle = \varepsilon/(1 - \varepsilon^2)^{5/2}$$

where the angular brackets denote a time average over one complete period.

**8-21.** Consider the family of orbits in a central potential for which the total energy is a constant. Show that if a stable circular orbit exists, the angular momentum associated with this orbit is larger than that for any other orbit of the family.

**8-22.** Discuss the motion of a particle moving in an attractive central-force field described by $F(r) = -k/r^3$.* Sketch some of the orbits for different values of the total energy. Can a circular orbit be stable in such a force field?

**8-23.** An Earth satellite moves in an elliptical orbit with a period $\tau$, eccentricity $\varepsilon$, and semimajor axis $a$. Show that the maximum radial velocity of the satellite is $2\pi a\varepsilon/(\tau\sqrt{1 - \varepsilon^2})$.

**8-24.** An Earth satellite has a perigee of 300 km and an apogee of 3,500 km above Earth's surface. How far is the satellite above Earth when **(a)** it has rotated 90° around Earth from perigee and **(b)** it has moved halfway from perigee to apogee?

**8-25.** An Earth satellite has a speed of 28,070 km/hr when it is at its perigee of 220 km above Earth's surface. Find the apogee distance, its speed at apogee, and its period of revolution.

**8-26.** Show that the most efficient way to change the energy of an elliptical orbit for a single short engine thrust is by firing the rocket along the direction of travel at perigee.

**8-27.** A spacecraft in an orbit about Earth has the speed of 10,160 m/s at a perigee of 6,680 km from Earth's center. What speed does the spacecraft have at apogee of 42,200 km?

**8-28.** What is the minimum escape velocity of a spacecraft from the moon?

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*This particular force law was extensively investigated by Roger Cotes (1682–1716), and the orbits are known as **Cotes' spirals.**

**8-29.** The minimum and maximum velocities of a moon rotating around Uranus are $v_{min} = v - v_0$ and $v_{max} = v + v_0$. Find the eccentricity in terms of $v$ and $v_0$.

**8-30.** A spacecraft is placed in orbit 200 km above Earth in a circular orbit. Calculate the minimum escape speed from Earth. Sketch the escape trajectory, showing Earth and the circular orbit. What is the spacecraft's trajectory with respect to Earth?

**8-31.** Consider a force law of the form

$$F(r) = -\frac{k}{r^2} - \frac{k'}{r^4}$$

Show that if $\rho^2 k > k'$, then a particle can move in a stable circular orbit at $r = \rho$.

**8-32.** Consider a force law of the form $F(r) = -(k/r^2)\exp(-r/a)$. Investigate the stability of circular orbits in this force field.

**8-33.** Consider a particle of mass $m$ constrained to move on the surface of a paraboloid whose equation (in cylindrical coordinates) is $r^2 = 4az$. If the particle is subject to a gravitational force, show that the frequency of small oscillations about a circular orbit with radius $\rho = \sqrt{4az_0}$ is

$$\omega = \sqrt{\frac{2g}{a + z_0}}$$

**8-34.** Consider the problem of the particle moving on the surface of a cone, as discussed in Examples 7.4 and 8.7. Show that the effective potential is

$$V(r) = \frac{l^2}{2mr^2} + mgr \cot \alpha$$

(Note that here $r$ is the radial distance in cylindrical coordinates, not spherical coordinates; see Figure 7-2.) Show that the turning points of the motion can be found from the solution of a cubic equation in $r$. Show further that only two of the roots are physically meaningful, so that the motion is confined to lie within two horizontal planes that cut the cone.

**8-35.** An almost circular orbit (i.e., $\varepsilon \ll 1$) can be considered to be a circular orbit to which a small perturbation has been applied. Then, the frequency of the radial motion is given by Equation 8.89. Consider a case in which the force law is $F(r) = -k/r^n$ (where $n$ is an integer), and show that the apsidal angle is $\pi/\sqrt{3 - n}$. Thus, show that a closed orbit generally results only for the harmonic oscillator force and the inverse-square-law force (if values of $n$ equal to or smaller than $-6$ are excluded).

**8-36.** A particle moves in an almost circular orbit in a force field described by $F(r) = -(k/r^2)\exp(-r/a)$. Show that the apsides advance by an amount approximately equal to $\pi\rho/a$ in each revolution, where $\rho$ is the radius of the circular orbit and where $\rho \ll a$.

**8-37.** A communication satellite is in a circular orbit around Earth at a distance above Earth equal to Earth's radius. Find the minimum velocity $\Delta v$ required to double the height of the satellite and put it in another circular orbit.

**8-38.** Calculate the minimum $\Delta v$ required to place a satellite already in Earth's heliocentric orbit (assumed circular) into the orbit of Venus (also assumed circular and coplanar with Earth). Consider only the gravitational attraction of the Sun. What time of flight would such a trip take?

**8-39.** Assuming a rocket engine can be fired only once from a low Earth orbit, does a Mars flyby or a Venus flyby require a larger $\Delta v$? Explain.

**8-40.** A spacecraft is being designed to dispose of nuclear waste either by carrying it out of the solar system or crashing into the Sun. Assume that no planetary flybys are permitted and that thrusts occur only in the orbital plane. Which mission requires the least energy? Explain.

**8-41.** A spacecraft is parked in a circular orbit 200 km above Earth's surface. We want to use a Hohmann transfer to send the spacecraft to the Moon's orbit. What are the total $\Delta v$ and the transfer time required?

**8-42.** A spacecraft of mass 10,000 kg is parked in a circular orbit 200 km above Earth's surface. What is the minimum energy required (neglect the fuel mass burned) to place the satellite in a synchronous orbit (i.e., $\tau = 24$ hr)?

**8-43.** A satellite is moving in circular orbit of radius $R$ about Earth. By what fraction must its velocity $v$ be increased for the satellite to be in an elliptical orbit with $r_{min} = R$ and $r_{max} = 2R$?

**8-44.** The Yukawa potential adds an exponential term to the long-range Coulomb potential, which greatly shortens the range of the Coulomb potential. It has great usefulness in atomic and nuclear calculations.

$$V(r) = \frac{V_0 r_0}{r} e^{-r/r_0} = -\frac{k}{r} e^{-r/a}$$

Find a particle's trajectory in a bound orbit of the Yukawa potential to first order in $r/a$.

**8-45.** A particle of mass $m$ moves in a central force field that has a constant magnitude $F_0$, but always points toward the origin. **(a)** Find the angular velocity $\omega_\phi$ required for the particle to move in a circular orbit of radius $r_0$. **(b)** Find the frequency $\omega_r$ of small radial oscillations about the circular orbit. Both answers should be in terms of $F_0$, $m$, and $r_0$.

**8-46.** Two double stars of the same mass as the sun rotate about their common center of mass. Their separation is 4 light years. What is their period of revolution?

**8-47.** Two double stars, one having mass 1.0 $M_{sun}$ and the other 3.0 $M_{sun}$, rotate about their common center of mass. Their separation is 6 light years. What is their period of revolution?