Notes 15 : UI Martingales

Math 733 - Fall 2013 *Lecturer: Sebastien Roch*

References: [Wil91, Chapter 13, 14], [Dur10, Section 5.5, 5.6, 5.7].

1 Uniform Integrability

We give a characterization of L^1 convergence. First note:

LEM 15.1 *Let $Y \in L^1$. $\forall \varepsilon > 0$, $\exists K > 0$ s.t.*

$$\mathbb{E}[|Y|; |Y| > K] < \varepsilon.$$

Proof: Immediate by (MON) to $\mathbb{E}[|Y|; |Y| \leq K]$. ■

What we need is for this condition to hold uniformly over the sequence:

DEF 15.2 (Uniform Integrability) *A collection $\mathcal{C}$ of RVs on $(\Omega, \mathcal{F}, \mathbb{P})$ is uniformly integrable (UI) if: $\forall \varepsilon > 0$, $\exists K > +\infty$ s.t.*

$$\mathbb{E}[|X|; |X| > K] < \varepsilon, \qquad \forall X \in \mathcal{C}.$$

THM 15.3 (Necessary and Sufficient Condition for L^1 Convergence) *Let $\{X_n\} \in L^1$ and $X \in L^1$. Then $X_n \to X$ in L^1 if and only if:*

- *$X_n \to X$ in prob*
- *$\{X_n\}$ is UI.*

Before giving the proof, we look at a few examples.

EX 15.4 (L^1-bddness is not sufficient) *Let $\mathcal{C}$ is UI and $X \in \mathcal{C}$. Note that*

$$\mathbb{E}|X| \leq \mathbb{E}[|X|; |X| \geq K] + \mathbb{E}[|X|; |X| < K] \leq \varepsilon + K < +\infty,$$

so UI implies L^1-bddness. But the opposite is not true by our last example (take $n^2 > K$).

EX 15.5 (L^p-bdd RVs) *But L^p-bddness works for $p > 1$. Let $\mathcal{C}$ be L^p-bdd and $X \in \mathcal{C}$. Then*

$$\mathbb{E}[|X|; |X| > K] \leq \mathbb{E}[K^{1-p}|X|^p; |X| > K|] \leq K^{1-p}A \to 0,$$

as $K \to +\infty$.

EX 15.6 (Dominated RVs) *Assume $\exists Y \in L^1$ s.t. $|X| \leq Y \; \forall X \in \mathcal{C}$. Then*

$$\mathbb{E}[|X|; |X| > K] \leq \mathbb{E}[Y; |X| > K] \leq \mathbb{E}[Y; Y > K],$$

and apply lemma above.

2 Proof of main theorem

Proof: We start with the if part. Fix $\varepsilon > 0$. We want to show that for n large enough:

$$\mathbb{E}|X_n - X| \leq \varepsilon.$$

It is natural to truncate at K to apply the UI property. Let $\phi_K(x) = \mathrm{sgn}(x)[|x| \wedge K]$. Then,

$$\begin{aligned}
\mathbb{E}|X_n - X| &\leq \mathbb{E}|\phi_K(X_n) - X_n| + \mathbb{E}|\phi_K(X) - X| + \mathbb{E}|\phi_K(X_n) - \phi_K(X)| \\
&\leq \mathbb{E}[|X_n|; |X_n| > K] + \mathbb{E}[|X|; |X| > K] + \mathbb{E}|\phi_K(X_n) - \phi_K(X)|.
\end{aligned}$$

The 1st term $\leq \varepsilon/3$ by UI and the 2nd term $\leq \varepsilon/3$ by lemma above. Check, by case analysis, that

$$|\phi_K(x) - \phi_K(y)| \leq |x - y|,$$

so $\phi_K(X_n) \to_P \phi_K(X)$. By bounded convergence for convergence in probability, the claim is proved.

LEM 15.7 (Bounded convergence theorem (convergence in probability version)) *Let $X_n \leq K < +\infty \; \forall n$ and $X_n \to_P X$. Then*

$$\mathbb{E}|X_n - X| \to 0.$$

Proof:(Sketch) By

$$\mathbb{P}[|X| \geq K + m^{-1}] \leq \mathbb{P}[|X_n - X| \geq m^{-1}],$$

it follows that $\mathbb{P}[|X| \leq K] = 1$. Fix $\varepsilon > 0$

$$\begin{aligned}
\mathbb{E}|X_n - X| &= \mathbb{E}[|X_n - X|; |X_n - X| > \varepsilon/2] + \mathbb{E}[|X_n - X|; |X_n - X| \leq \varepsilon/2] \\
&\leq 2K\mathbb{P}[|X_n - X| > \varepsilon/2] + \varepsilon/2 < \varepsilon,
\end{aligned}$$

for n large enough. ∎

Proof of only if part. Suppose $X_n \to X$ in L^1. We know that L^1 implies convergence in probability. So the first claim follows.

For the second claim, if $n \geq N$ (large enough),

$$\mathbb{E}|X_n - X| \leq \varepsilon.$$

We can choose K large enough so that

$$\mathbb{E}[|X_n|; |X_n| > K] < \varepsilon,$$

$\forall n \leq N$. (Because there is only a finite number.) So only need to worry about $n > N$. To use L^1 convergence, natural to write

$$\mathbb{E}[|X_n|; |X_n| > K] \leq \mathbb{E}[|X_n - X|; |X_n| > K] + \mathbb{E}[|X|; |X_n| > K].$$

First term $\leq \varepsilon$. The issue with the second term is that we cannot apply the lemma because the event involves X_n rather than X. In fact, a stronger version exists:

LEM 15.8 (Absolute continuity) *Let $X \in L^1$. $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t. $\mathbb{P}[F] < \delta$ implies*

$$\mathbb{E}[|X|; F] < \varepsilon.$$

Proof: Argue by contradiction. Suppose there is ε and F_n s.t. $\mathbb{P}[F_n] \leq 2^{-n}$ and

$$\mathbb{E}[|X|; F_n] \geq \varepsilon.$$

By BC,

$$\mathbb{P}[H] \equiv \mathbb{P}[F_n \text{ i.o.}] = 0.$$

By reverse Fatou (applied to $|X|\mathbb{1}_H = \limsup |X|\mathbb{1}_{F_n}$),

$$\mathbb{E}[|X|; H] \geq \varepsilon,$$

a contradiction. ∎

To conclude note that

$$\mathbb{P}[|X_n| > K] \leq \frac{\mathbb{E}|X_n|}{K} \leq \frac{\sup_{n \geq N} \mathbb{E}|X_n|}{K} \leq \frac{\sup_{n \geq N} \mathbb{E}|X| + \mathbb{E}|X_n - X|}{K} < \delta,$$

uniformly in n for K large enough. We are done.

∎

3 UI MGs

THM 15.9 (Convergence of UI MGs) *Let M be UI MG. Then*

$$M_n \to M_\infty,$$

a.s. and in L^1. Moreover,

$$M_n = \mathbb{E}[M_\infty \,|\, \mathcal{F}_n], \qquad \forall n.$$

Proof: UI implies L^1-bddness so we have $M_n \to M_\infty$ a.s. By necessary and sufficient condition, we also have L^1 convergence.

Now note that for all $r \geq n$ and $F \in \mathcal{F}_n$, we know $\mathbb{E}[M_r \,|\, \mathcal{F}_n] = M_n$ or

$$\mathbb{E}[M_r; F] = \mathbb{E}[M_n; F],$$

by definition of CE. We can take a limit by L^1 convergence. More precisely

$$|\mathbb{E}[M_r; F] - \mathbb{E}[M_\infty; F]| \leq \mathbb{E}[|M_r - M_\infty|; F] \leq \mathbb{E}[|M_r - M_\infty|] \to 0,$$

as $r \to \infty$. So plugging above

$$\mathbb{E}[M_\infty; F] = \mathbb{E}[M_n; F],$$

and $\mathbb{E}[M_\infty \,|\, \mathcal{F}_n] = M_n$. ■

4 Applications I

THM 15.10 (Levy's upward thm) *Let $Z \in L^1$ and define $M_n = \mathbb{E}[Z \,|\, \mathcal{F}_n]$. Then M is a UI MG and*

$$M_n \to M_\infty = \mathbb{E}[Z \,|\, \mathcal{F}_\infty],$$

a.s. and in L^1.

Proof: M is a MG by (TOWER). We first show it is UI:

LEM 15.11 *Let $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$. Then*

$$\{\mathbb{E}[X \,|\, \mathcal{G}] \,:\, \mathcal{G} \text{ is a sub-}\sigma\text{-field of } \mathcal{F}\},$$

is UI.

Proof: We use the absolute continuity lemma again. Let $Y = \mathbb{E}[X \,|\, \mathcal{G}] \in \mathcal{G}$. Since $\{|Y| > K\} \in \mathcal{G}$,

$$\begin{aligned} \mathbb{E}[|Y|; |Y| > K] &= \mathbb{E}[|\mathbb{E}[X \,|\, \mathcal{G}]|; |Y| > K] \\ &\leq \mathbb{E}[\mathbb{E}[|X| \,|\, \mathcal{G}]; |Y| > K] \\ &= \mathbb{E}[|X|; |Y| > K]. \end{aligned}$$

By Markov and (JENSEN)

$$\mathbb{P}[|Y| > K] \leq \frac{\mathbb{E}|Y|}{K} \leq \frac{\mathbb{E}|X|}{K} \leq \delta,$$

for K large enough (uniformly in $\mathcal{G}$). And we are done. ■

In particular, we have convergence a.s. and in L^1 to $M_\infty \in \mathcal{F}_\infty$.

Let $Y = \mathbb{E}[Z \,|\, \mathcal{F}_\infty] \in \mathcal{F}_\infty$. By dividing into negative and positive parts, we assume $Z \geq 0$. We want to show, for $F \in \mathcal{F}_\infty$,

$$\mathbb{E}[Z; F] = \mathbb{E}[M_\infty; F].$$

By Uniqueness Lemma, it suffices to prove equality for all $\mathcal{F}_n$. If $F \in \mathcal{F}_n \subseteq \mathcal{F}_\infty$, then by (TOWER)

$$\mathbb{E}[Z; F] = \mathbb{E}[Y; F] = \mathbb{E}[M_n; F] = \mathbb{E}[M_\infty; F].$$

The first equality is by definition of Y; the second equality is by definition of M_n; the third equality is from our main theorem. ■

THM 15.12 (Levy's $0-1$ law) *Let $A \in \mathcal{F}_\infty$. Then*

$$\mathbb{P}[A \,|\, \mathcal{F}_n] \to \mathbb{1}_A.$$

Proof: Immediate. ■

COR 15.13 (Kolmogorov's $0-1$ law) *Let $X_1, X_2, \ldots$ be iid RVs. Recall that the tail σ-field is*

$$\mathcal{T} = \cap_n \mathcal{T}_n = \cap_n \sigma(X_{n+1}, X_{n+2}, \ldots).$$

If $A \in \mathcal{T}$ then $\mathbb{P}[A] \in \{0, 1\}$.

Proof: Since $A \in \mathcal{T}_n$ is independent of $\mathcal{F}_n$,

$$\mathbb{P}[A \,|\, \mathcal{F}_n] = \mathbb{P}[A],$$

$\forall n$. By Levy's law,

$$\mathbb{P}[A] = \mathbb{1}_A \in \{0, 1\}.$$

■

5 Applications II

THM 15.14 (Levy's Downward Thm) *Let* $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ *and* $\{\mathcal{G}_{-n}\}_{n\geq 0}$ *a collection of* σ*-fields s.t.*

$$\mathcal{G}_{-\infty} = \cap_k \mathcal{G}_{-k} \subseteq \cdots \subseteq \mathcal{G}_{-n} \subseteq \cdots \subseteq \mathcal{G}_{-1} \subseteq \mathcal{F}.$$

Define

$$M_{-n} = \mathbb{E}[Z \,|\, \mathcal{G}_{-n}].$$

Then

$$M_{-n} \to M_{-\infty} = \mathbb{E}[Z \,|\, \mathcal{G}_{-\infty}]$$

a.s. and in L^1*.*

Proof: We apply the same argument as in the Martingale Convergence Thm. Let $\alpha < \beta \in \mathbb{Q}$ and

$$\Lambda_{\alpha,\beta} = \{\omega \,:\, \liminf X_{-n} < \alpha < \beta < \limsup X_{-n}\}.$$

Note that

$$\begin{aligned} \Lambda &\equiv \{\omega \,:\, X_n \text{ does not converge}\} \\ &= \{\omega \,:\, \liminf X_{-n} < \limsup X_{-n}\} \\ &= \cup_{\alpha<\beta\in\mathbb{Q}} \Lambda_{\alpha,\beta}. \end{aligned}$$

Let $U_N[\alpha, \beta]$ be the *number of upcrossings of* $[\alpha, \beta]$ *between time* $-N$ *and* -1. Then by the Upcrossing Lemma applied to the MG $M_{-N}, \ldots, M_{-1}$

$$(\beta - \alpha)\mathbb{E}U_N[\alpha, \beta] \leq |\alpha| + \mathbb{E}|M_{-1}| \leq |\alpha| + \mathbb{E}|Z|.$$

By (MON)

$$U_N[\alpha, \beta] \uparrow U_\infty[\alpha, \beta],$$

and

$$(\beta - \alpha)\mathbb{E}U_\infty[\alpha, \beta] \leq |\alpha| + \mathbb{E}|Z| < +\infty,$$

so that

$$\mathbb{P}[U_\infty[\alpha, \beta] = \infty] = 0.$$

Since

$$\Lambda_{\alpha,\beta} \subseteq \{U_\infty[\alpha, \beta] = \infty\},$$

we have $\mathbb{P}[\Lambda_{\alpha,\beta}] = 0$. By countability, $\mathbb{P}[\Lambda] = 0$. Therefore we have convergence a.s.

By lemma in previous class, M is UI and hence we have L^1 convergence as well.

Finally, for all $G \in \mathcal{G}_{-\infty} \subseteq \mathcal{G}_{-n}$,

$$\mathbb{E}[Z; G] = \mathbb{E}[M_{-n}; G].$$

Take the limit $n \to +\infty$ and use L^1 convergence. ■

5.1 Law of large numbers

An application:

THM 15.15 (Strong Law; Martingale Proof) *Let $X_1, X_2, \ldots$ be iid RVs with $\mathbb{E}[X_1] = \mu$ and $\mathbb{E}|X_1| < +\infty$. Let $S_n = \sum_{i \leq n} X_n$. Then*

$$n^{-1} S_n \to \mu,$$

a.s. and in L^1.

Proof: Let

$$\mathcal{G}_{-n} = \sigma(S_n, S_{n+1}, S_{n+2}, \ldots) = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots),$$

and note that, for $1 \leq i \leq n$,

$$\mathbb{E}[X_1 \,|\, \mathcal{G}_{-n}] = \mathbb{E}[X_1 \,|\, S_n] = \mathbb{E}[X_i \,|\, S_n] = \mathbb{E}[n^{-1} S_n \,|\, S_n] = n^{-1} S_n,$$

by symmetry. By Levy's Downward Thm

$$n^{-1} S_n \to \mathbb{E}[X_1 \,|\, \mathcal{G}_{-\infty}],$$

a.s. and in L^1. But the limit must be trivial by Kolmogorov's 0-1 law and we must have $\mathbb{E}[X_1 \,|\, \mathcal{G}_{-\infty}] = \mu$. ■

5.2 Hewitt-Savage

DEF 15.16 *Let $X_1, X_2, \ldots$ be iid RVs. Let $\mathcal{E}_n$ be the σ-field generated by events invariant under permutations of the Xs that leave $X_{n+1}, X_{n+2}, \ldots$ unchanged. The* exchangeable σ-field *is $\mathcal{E} = \cap_m \mathcal{E}_m$.*

THM 15.17 (Hewitt-Savage 0-1 law) *Let $X_1, X_2, \ldots$ be iid RVs. If $A \in \mathcal{E}$ then $\mathbb{P}[A] \in \{0, 1\}$.*

Proof: The idea of the proof is to show that A is independent of itself. Indeed, we then have

$$0 = \mathbb{P}[A] - \mathbb{P}[A \cap A] = \mathbb{P}[A] - \mathbb{P}[A]\mathbb{P}[A] = \mathbb{P}[A](1 - \mathbb{P}[A]).$$

Since $A \in \mathcal{E}$ and $A \in \mathcal{F}_\infty$, it suffices to show that $\mathcal{E}$ is independent of $\mathcal{F}_n$ for every n (by the π-λ theorem).

WTS: for every bounded ϕ, $B \in \mathcal{E}$,

$$\mathbb{E}[\phi(X_1, \ldots, X_k); B] = \mathbb{E}[\phi(X_1, \ldots, X_k)]\mathbb{E}[B] = \mathbb{E}[\mathbb{E}[\phi(X_1, \ldots, X_k)]; B],$$

or equivalently

$$Y = \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}] = \mathbb{E}[\phi(X_1, \ldots, X_k)].$$

It suffices to show that Y is independent of $\mathcal{F}_k$. Indeed, by the L^2 characterization of conditional expectation and independence,

$$0 = \mathbb{E}[(\phi(X_1, \ldots, X_k) - Y)Y] = \mathbb{E}[\phi(X_1, \ldots, X_k)]\mathbb{E}[Y] - \mathbb{E}[Y^2] = -\mathrm{Var}[Y],$$

and Y is constant.

1. Since ϕ is bounded, it is integrable and Levy's Downward Thm implies
$$\mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}_n] \to \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}].$$

2. We make ϕ "exchangeable" by averaging over all configurations and taking a limit as $n \to +\infty$. Define
$$A_n(\phi) = \frac{1}{(n)_k} \sum_{1 \leq i_1 \neq \cdots \neq i_k \leq n} \phi(X_{i_1}, \ldots, X_{i_k}),$$
where $(n)_k = n(n-1)\cdots(n-k+1)$. Note by symmetry
$$A_n(\phi) = \mathbb{E}[A_n(\phi) \,|\, \mathcal{E}_n] = \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}_n] \to \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}].$$

3. The reason we did this is that now the first k Xs have little influence on this quantity and therefore the limit is independent of them. However, note that
$$\frac{1}{(n)_k} \sum_{1 \in \mathbf{i}} \phi(X_{i_1}, \ldots, X_{i_k}) \leq \frac{k(n-1)_{k-1}}{(n)_k} \sup \phi = \frac{k}{n} \sup \phi \to 0,$$
so that the limit of $A_n(\phi)$ is independent of X_1 and
$$\mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}] \in \sigma(X_2, \ldots),$$
and by induction
$$Y = \mathbb{E}[\phi(X_1, \ldots, X_k) \,|\, \mathcal{E}] \in \sigma(X_{k+1}, \ldots).$$

■

6 Optional Sampling

6.1 Review: Stopping times

Recall:

DEF 15.18 *A random variable $T : \Omega \to \overline{\mathbb{Z}}_+ \equiv \{0, 1, \ldots, +\infty\}$ is called a* stopping time *if*

$$\{T = n\} \in \mathcal{F}_n, \ \forall n \in \overline{\mathbb{Z}}_+.$$

EX 15.19 *Let $\{A_n\}$ be an adapted process and $B \in \mathcal{B}$. Then*

$$T = \inf\{n \geq 0 \, : \, A_n \in B\},$$

is a stopping time.

THM 15.20 *Let $\{M_n\}$ be a MG and T be a stopping time. Then M_T is integrable and*

$$\mathbb{E}[M_T] = \mathbb{E}[X_0].$$

if one of the following holds:

1. *T is bounded.*
2. *M is bounded and T is a.s. finite.*
3. *$\mathbb{E}[T] < +\infty$ and M has bounded increments.*
4. *M is UI. (This one is new. The proof follows from the Optional Sampling Theorem below.)*

DEF 15.21 ($\mathcal{F}_T$) *Let T be a stopping time. Denote by $\mathcal{F}_T$ the set of all events F such that $\forall n \in \overline{\mathbb{Z}}_+$*

$$F \cap \{T = n\} \in \mathcal{F}_n.$$

6.2 More on the σ-field $\mathcal{F}_T$

The following two lemmas help clarify the definition of $\mathcal{F}_T$:

LEM 15.22 *$\mathcal{F}_T = \mathcal{F}_n$ if $T \equiv n$, $\mathcal{F}_T = \mathcal{F}_\infty$ if $T \equiv \infty$ and $\mathcal{F}_T \subseteq \mathcal{F}_\infty$ for any T.*

Proof: In the first case, note $F \cap \{T = k\}$ is empty if $k \neq n$ and is F if $k = n$. So if $F \in \mathcal{F}_T$ then $F = F \cap \{T = n\} \in \mathcal{F}_n$ and if $F \in F_n$ then $F = F \cap \{T = n\} \in F_n$. Moreover $\emptyset \in \mathcal{F}_n$ so we have proved both inclusions. This works also for $n = \infty$. For the third claim note

$$F = \cup_{k \in \overline{\mathbb{Z}}_+} F \cap \{T = n\} \in \mathcal{F}_\infty.$$

■

LEM 15.23 *If X is adapted and T is a stopping time then $X_T \in \mathcal{F}_T$ (where we assume that $X_\infty \in \mathcal{F}_\infty$, e.g., $X_\infty = \liminf X_n$).*

Proof: For $B \in \mathcal{B}$

$$\{X_T \in B\} \cap \{T = n\} = \{X_n \in B\} \cap \{T = n\} \in \mathcal{F}_n.$$

■

LEM 15.24 *If S, T are stopping times then $\mathcal{F}_{S \wedge T} \subseteq \mathcal{F}_T$.*

Proof: Let $F \in \mathcal{F}_{S \wedge T}$. Note that

$$F \cap \{T = n\} = \cup_{k \leq n} [(F \cap \{S \wedge T = k\}) \cap \{T = n\}] \in \mathcal{F}_n.$$

Indeed, the expression in parenthesis is in $\mathcal{F}_k \subseteq \mathcal{F}_n$ and $\{T = n\} \in \mathcal{F}_n$. ■

6.3 Optional Sampling Theorem (OST)

THM 15.25 (Optional Sampling Theorem) *If M is a UI MG and S, T are stopping times with $S \leq T$ a.s. then $\mathbb{E}|M_T| < +\infty$ and*

$$\mathbb{E}[M_T \,|\, \mathcal{F}_S] = M_S.$$

Proof: Since M is UI, $\exists M_\infty \in \mathcal{L}^1$ s.t. $M_n \to M_\infty$ a.s. and in $\mathcal{L}^1$. We prove a more general claim:

LEM 15.26

$$\mathbb{E}[M_\infty \,|\, \mathcal{F}_T] = M_T.$$

Indeed, we then get the theorem by (TOWER) and (JENSEN).
Proof:(Lemma) Wlog we assume $M_\infty \geq 0$ so that $M_n = \mathbb{E}[M_\infty \,|\, \mathcal{F}_n] \geq 0$ $\forall n$. Let $F \in \mathcal{F}_T$. Then (trivially)

$$\mathbb{E}[M_\infty; F \cap \{T = \infty\}] = \mathbb{E}[M_T; F \cap \{T = \infty\}]$$

so STS
$$\mathbb{E}[M_\infty; F \cap \{T < +\infty\}] = \mathbb{E}[M_T; F \cap \{T < +\infty\}].$$
In fact, by (MON), STS
$$\mathbb{E}[M_\infty; F \cap \{T \leq k\}] = \mathbb{E}[M_T; F \cap \{T \leq k\}] = \mathbb{E}[M_{T\wedge k}; F \cap \{T \leq k\}],$$
$\forall k$. To conclude we make two observations:

1. $\boxed{F \cap \{T \leq k\} \in \mathcal{F}_{T\wedge k}.}$ Indeed if $n \leq k$
$$F \cap \{T \leq k\} \cap \{T \wedge k = n\} = F \cap \{T = n\} \in \mathcal{F}_n,$$
and if $n > k$
$$= \emptyset \in \mathcal{F}_n.$$

2. $\boxed{\mathbb{E}[M_\infty \,|\, \mathcal{F}_{T\wedge k}] = M_{T\wedge k}.}$ Since $\mathbb{E}[M_\infty \,|\, \mathcal{F}_k] = M_k$, STS $\mathbb{E}[M_k \,|\, \mathcal{F}_{T\wedge k}] = M_{T\wedge k}$. But note that if $G \in \mathcal{F}_{T\wedge k}$
$$\mathbb{E}[M_k; G] = \sum_{l\leq k} \mathbb{E}[M_k; G\cap\{T\wedge k = l\}] = \sum_{l\leq k} \mathbb{E}[M_l; G\cap\{T\wedge k = l\}] = \mathbb{E}[M_{T\wedge k}; G]$$
since $G \cap \{T \wedge k = l\} \in \mathcal{F}_l$.

■ ■

6.4 Example: Biased RW

DEF 15.27 *The* asymmetric simple RW *with parameter* $1/2 < p < 1$ *is the process* $\{S_n\}_{n\geq 0}$ *with* $S_0 = 0$ *and* $S_n = \sum_{k\leq n} X_k$ *where the* X_k*s are iid in* $\{-1, +1\}$ *s.t.* $\mathbb{P}[X_1 = 1] = p$. *Let* $q = 1 - p$. *Let* $\phi(x) = (q/p)^x$ *and* $\psi_n(x) = x - (p - q)n$.

THM 15.28 *Let* $\{S_n\}$ *as above. Let* $a < 0 < b$. *Define* $T_x = \inf\{n \geq 0 \,:\, S_n = x\}$. *Then*

1. *We have*
$$\mathbb{P}[T_a < T_b] = \frac{\phi(b) - \phi(0)}{\phi(b) - \phi(a)}.$$
In particular, $\mathbb{P}[T_a < +\infty] = 1/\phi(a)$ *and* $\mathbb{P}[T_b < \infty] = 1$.

2. *We have*
$$\mathbb{E}[T_b] = \frac{b}{2p - 1}.$$

Proof: There are two MGs here:

$$\mathbb{E}[\phi(S_n) \mid \mathcal{F}_{n-1}] = p(q/p)^{S_{n-1}+1} + q(q/p)^{S_{n-1}-1} = \phi(S_{n-1}),$$

and

$$\mathbb{E}[\psi_n(S_n) \mid \mathcal{F}_{n-1}] = p[S_{n-1}+1-(p-q)(n)]+q[S_{n-1}-1-(p-q)(n)] = \psi_{n-1}(S_{n-1}).$$

Let $N = T_a \wedge T_b$. Now note that $\phi(S_{N\wedge n})$ is a bounded MG and therefore applying the MG property at time n and taking limits as $n \to \infty$ (using (DOM))

$$\phi(0) = \mathbb{E}[\phi(S_N)] = \mathbb{P}[T_a < T_b]\phi(a) + \mathbb{P}[T_a > T_b]\phi(b),$$

where we need to prove that $N < +\infty$ a.s. Indeed, since $(b-a)$ $+1$-steps always take us out of (a, b),

$$\mathbb{P}[T_b > n(b-a)] \leq (1 - q^{b-a})^n,$$

so that

$$\mathbb{E}[T_b] = \sum_{k\geq 0} \mathbb{P}[T_b > k] \leq \sum_n (b-a)(1-q^{b-a})^n < +\infty.$$

In particular $T_b < +\infty$ a.s. and $N < +\infty$ a.s. Rearranging the formula above gives the first result. (For the second part of the first result, take $b \to +\infty$ and use monotonicity.)

For the third one, note that $T_b \wedge n$ is bounded so that

$$0 = \mathbb{E}[S_{T_b\wedge n} - (p-q)(T_b \wedge n)].$$

By (MON), $\mathbb{E}[T_b \wedge n] \uparrow \mathbb{E}[T_b]$. Finally, using

$$\mathbb{P}[-\inf_n S_n \geq -a] = \mathbb{P}[T_a < +\infty],$$

and the fact that $-\inf_n S_n \geq 0$ shows that $\mathbb{E}[-\inf_n S_n] < +\infty$. Hence, we can use (DOM) with $|S_{T_b\wedge n}| \leq \max\{b, -\inf_n S_n\}$.

■

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