A Note on Potential Functions, Congestion Games, and the Shapley Value

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Abstract. We consider a large class of non-cooperative games that includes most known variations of congestion games. We show that if we divide the utility in such a generalized congestion game according to the [weighted] Shapley values of the agents using each resource, then every such game has an exact [weighted] potential function. This is by applying a classic result by Hart and Mas-Colell [3] on exact and weighted potential functions of cooperative games with transferable utility. We then show how recent results on the existence of potential functions in various classes of non-cooperative games follow as a special case from this observation.

1 Introduction

Congestion games are a common and useful representation of non-cooperative games. Many variations and extensions of congestion games have been proposed, so as to encompass a large variety of settings, interactions and applications. Variations differ in how the total generated utility (or cost) on each resource is affected by the identity and actions of players, as well as how it is being divided among them.

While it is well known that "standard" congestion games have a potential function and thus guarantee convergence of best-response strategies to a Nash equilibrium, this is not true for most extensions that have been suggested for congestion games. For example, congestion games where players are weighted may not even have a pure Nash equilibrium.

We describe *Generalized Congestion Games* (GCG), where the total utility function of each resource is unrestricted, and may be divided arbitrarily. Through a simple observation that builds on a result by Hart and Mas-Colell [3], we show that every GCG has an exact potential function, provided that the total utility of each resource is being divided according to the Shapley values of users (i.e. each player's expected marginal contribution in a random order). Similarly, if utility is divided according to some weighted Shapley value, then the game admits a weighted potential function (w.r.t. the same weights).

Some recent results in the literature about the existence of potential functions in congestion games (namely, [5, 6]) follow as special cases from this observation. Moreover, we show that these results can be strengthened in some cases.

2 Background and Preliminaries

2.1 Cooperative games

A transferable utility (TU) cooperative game is a pair (N, v), where N is a finite set of agents, and $v : 2^N \to \mathbb{R}_+$ is a value function (also called the characteristic function).

Given a TU game (N, v), a *preimputation* $\mathbf{x}$ is a division of $v(N)$ between the members of N. An *efficient point solution concept* is a mapping from every TU game (N, v) to a preimputation $\psi(N, v) = (\psi_1, \ldots, \psi_n)$.

Unanimity games For every $\emptyset \neq T \subseteq N$, the (TU) unanimity game u_T is defined as $u_T(S) = 1$ if $S \supseteq T$ and 0 otherwise. Every TU game (N, v) has a unique decomposition as $v = \sum_{\emptyset \neq T \subseteq N} \alpha_T u_T$.

The Shapley value The *Shapley value* $\phi(N, v)$ is an efficient point solution concept defined as the average marginal contribution of agent i when agents are ordered randomly [11]. Formally,

$$\phi^i(N, v) = \frac{1}{n!} \sum_{\pi \in Sym(n)} \left(v(S^i(\pi) \cup \{i\}) - v(S^i(\pi)) \right).$$

The Shapley value of i in the unanimity game u_T is $1/|T|$ if $i \in T$ and 0 otherwise. Since the Shapley value is additive, it holds that

$$\phi^i(N, v) = \sum_{\emptyset \neq T \subseteq N} \alpha_T \phi^i(u_T) = \sum_{T \subseteq N : i \in T} \frac{\alpha_T}{|T|}, \tag{1}$$

where each α_T is the coefficient of u_T in the unique decomposition of (N, v).

Weighted Shapley values Given weights $\mathbf{w} = (w_1, \ldots, w_n)$, the *weighted Shapley value* [4] of i in the unanimity game (N, u_T) is $\phi^i_{\mathbf{w}}(N, u_T) = \frac{w_i}{\sum_{j \in T} w_j}$ for $i \in T$ and 0 for $i \notin T$. As with the standard Shapley values, it is extended to general games as follows:

$$\phi^i_{\mathbf{w}}(N, v) = \sum_{T \subseteq N} \alpha_T \phi^i_{\mathbf{w}}(u_T) = \sum_{T \subseteq N, i \in T} \alpha_T \frac{w_i}{\sum_{j \in T} w_j},$$

where $v = \sum_T \alpha_T u_T$ is the decomposition of (N, v).

The HMC potential of TU games Hart and Mas-Colell [3] define the potential of every TU game $P(N, v)$, and its differentials: $D^i P(N, v) = P(N, v) - P(N \setminus \{i\}, v)$. They require that for any game (N, v),

$$\sum_{i \in N} D^i P(N, v) = v(N). \tag{2}$$

They prove that such a potential function exists and its differentials are equal to the Shapley value, i.e. $D^i P(N,v) = \phi^i(N,v)$.

Hart and Mas-Collel then prove a more general result. For every vector of weights $\mathbf{w}$, they say that $P_{\mathbf{w}}(N,v)$ is a *weighted potential* function if it holds that $\sum_{i\in N} w_i D^i P_{\mathbf{w}}(N,v) = v(N)$. They show that for any $\mathbf{w}$, such functions exist, where $D^i P_{\mathbf{w}}(N,v) = \phi^i_{\mathbf{w}}(N,v)$.

We refer to the functions P and $P_{\mathbf{w}}$ as the *HMC potential* and the *weighted HMC potential*, respectively.

2.2 Congestion Games and Potential Games

Potential functions A *non-cooperative game* G with n agents has strategy sets $(A_i)_{i\in N}$, and n utility functions $u_i : \mathbf{A} \to \mathbb{R}$, where $\mathbf{A} = \times_{i\in N} A_i$. A *potential function* of G is a function $\Phi : \mathbf{A} \to \mathbb{R}$, s.t. for every $\mathbf{a} \in \mathbf{A}$, agent j, and $a'_j \in A_j$,

$$\Phi(\mathbf{a}) - \Phi(a_{-j}, a'_j) = u_j(\mathbf{a}) - u_j(a_{-j}, a'_j).$$

Given a weight vector $\mathbf{w} = (w_1, \ldots, w_n)$, we say that Φ is a $\mathbf{w}$-*weighted potential function* of G if for every $\mathbf{a} \in \mathbf{A}$, agent j, and $a'_j \in A_j$,

$$\Phi(\mathbf{a}) - \Phi(a_{-j}, a'_j) = w_j(u_j(\mathbf{a}) - u_j(a_{-j}, a'_j)).$$

Generalized Congestion games A Generalized Congestion Game (GCG), $G = (N, F, (A_i)_{i\in N}, (U_x)_{x\in F}, (u^i_x)^{i\in N}_{x\in F})$, has a set of agents N, and a set of resources F. Every agent is restricted to select from some $A_i \subseteq 2^F$. That is, a strategy of i is a set of resources $a_i \in A_i$. for example, F can be edges in a graph, and A_i are all paths from the home of agent i (some vertex s_i) to target vertex t. In a particular profile $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbf{A}$, we denote by $N_x(\mathbf{a})$ (or just N_x when the profile is clear from the context) the set of agents using resource x. For every $x \in F$ there is a utility function $U_x : 2^N \to \mathbb{R}$, and a method to divide the utility among the agents of N_x to $(u^i_x(N_x))_{i\in N_x}$, s.t. $\sum_{i\in N_x} u^i_x(N_x) = U_x(N_x)$. The total utility of player i under profile $\mathbf{a}$ is $u^i(\mathbf{a}) = \sum_{x\in a_i} u^i_x(N_x(\mathbf{a}))$. Thus a GCG is a non-cooperative game.

While we are not aware of a previous definition of GCG, several special cases have been studied in the literature. If U_x only depends on $|N_x|$, and $u^i_x(|N_x|) = \frac{1}{|N_x|} U_x(|N_x|)$ for all x and i, then F is called a *congestion game* [9]. A *cost sharing game* is a congestion game where a fixed *cost* is equally divided among players using a resource. That is, $U_x(N_x) = -\gamma_x$ is a negative constant. Rosenthal [9] showed that congestion games always possess an exact potential function. Later, Monderer and Shapley proved that the converse is also true. That is, G is (isomorphic to) a congestion game if and only if G has a potential function [8]. They also include the following sentence (our emphasis):

*"Hart and Mas-Colell (1989) have applied potential theory to cooperative games. Except for the fact that we are all using potential theory our works are **not connected**."*

However, this connection turned out to be quite strong. A preliminary connection was shown already in the paper of Monderer and Shapley (see Lemma 1). Later, Ui [13] proposed an alternative characterization for (non-weighted) potential games, by showing that a game admits a potential if and only if the payoff functions coincide with the Shapley value of a particular class of cooperative games. In what follows, we show how this interesting connection can be further exploited to reveal potentials in various games.

Other special cases of GCG in the literature Two generalizations of congestion games were presented by Milchtaich [7], both of which are special cases of GCG. The first is *Player specific congestion game*, where each U_x depends only on $|N_x|$, but can be divided arbitrarily among players. The second is *weighted congestion games*, where each player has an intrinsic weight $\beta_i \in \mathbb{R}_+$, and every $U_x : \mathbb{R}_+ \to \mathbb{R}$ is a function $\beta(N_x) = \sum_{i \in N_x} \beta_i$. Traditionally utility is distributed proportionally to the weight, i.e. where $u^i_x(N_x) = \frac{\beta_i}{\beta(N_x)} U_x(N_x)$. He also assumes that u^i_x is monotonically non-increasing.

Milchtaich showed that games in neither class possess a potential function, although player specific games do have a pure Nash equilibrium. Also, in some special cases an exact potential function still exits.

Kollias and Roughgarden [5] highlight two particular cases where a variant of weighted congestion games has an exact or weighted potential. The first result states that if we take monotone weighted congestion games and divide the total cost (negative utility) $U_x(N_x)$ according to the *Shapley values* of players in N_x (rather than proportionally to their weights, as in [7]), then the game admits an exact potential function. The second result considers a variant of cost sharing games, augmented with some weight vector $\boldsymbol{\beta}$. Here the authors show that if the total cost γ_x is shared according to the *weighted Shapley values* $\phi_{\boldsymbol{\beta}}$ of the players using x, then the game admits a $\boldsymbol{\beta}$-weighted potential function. Note that both variants are special cases of GCG.

In a recent paper, Meir et al. [6] introduced the concept of agent failures in congestion games, focusing on games with costs (negative utilities). In their model, an original congestion game G is perturbed by randomly selecting a set of surviving players $S \subseteq N$, according to some distribution $\mathbf{p}$, where $\sum_{S \subseteq N} p(S) = 1$. The cost for every player in the perturbed game is her expected cost over all instantiations of the game. Since the expected total cost is well defined, this is a GCG. The authors show that every such game admits a weighted potential function.

3 Results

Monderer and Shapley [8] show in their paper a modest version of the connection between TU games and congestion games, linking the HMC potential to the potential of non-cooperative "participation games".

For any TU game (N, v) + point solution concept ψ, Monderer and Shapley define the induced non-cooperative "participation game" $\Gamma(v, \psi)$. That is, $u_i(\mathbf{a}) = \psi^i(N_{\mathbf{a}}, v)$, where $N_{\mathbf{a}}$ is that set of agents that chose to participate.

Lemma 1 (Monderer and Shapley [8]). *(a) Γ has an exact potential function iff ψ is the Shapley value. (b) Γ has a weighted potential function iff ψ is the weighted Shapley value for the same weight vector (with the same weights).*

As expected, Monderer and Shapley prove the lemma by mapping the participation game to a TU game, and applying the [weighted] HMC potential function $P_{\mathbf{w}}$. We are now ready to present our results.

Theorem 1. *Let G be a GCG.*

1. *If the total utility U_x on every $x \in F$ is divided by the Shapley value, i.e. if $u^i_x(N_x) = \phi^i(N_x, U_x)$ for all x, i, then G has an exact potential function.*
2. *If the total utility U_x on every $x \in F$ is divided by a weighted Shapley value, i.e. if $u^i_x(N_x) = \phi^i_{\mathbf{w}}(N_x, U_x)$ for all x, i, then G has a $\mathbf{w}$-potential function.*

Proof. It is sufficient to prove the second part, as the first part is a special case for equal weights. Every resource $x \in F$ induces a participation game $\Gamma_x = \Gamma(U_x, \phi)$, where the set of "participating" agents in Γ_x is the set N_x (players who use resource x). Thus by Lemma 1(b), Γ_x has a $\mathbf{w}$-potential function $P_{\mathbf{w}}(N_x, U_x)$. We define the function

$$\Phi(\mathbf{a}) = \Phi(N_1, \ldots, N_{|F|}) = \sum_{x \in F} P_{\mathbf{w}}(N_x, U_x),$$

and argue that it is a $\mathbf{w}$-weighted potential function of G.

Indeed, suppose that some agent i switches from resources $a_i \subseteq F$ to $a'_i \subseteq F$. W.l.o.g. a_i, a'_i are disjoint, as resources in the intersection do not change the difference in utility or potential.

$$\begin{aligned}
w_i(u_i(a_i) - u_i(a'_i)) &= w_i \sum_{x \in a_i} u_{x,i}(N_x) + w_i \sum_{x \in a'_i} u_{x,i}(N_x) \\
&\quad - w_i \left(\sum_{x \in a_i} u_{x,i}(N_x \setminus \{i\}) + \sum_{x \in a'_i} u_{x,i}(N_x \cup \{i\}) \right) \\
&= \sum_{x \in a_i} w_i(u_{x,i}(N_x) - u_{x,i}(N_x \setminus \{i\})) - \sum_{x \in a'_i} w_i(u_{x,i}(N_x \cup \{i\}) - u_{x,i}(N_x)) \\
&= \sum_{x \in a_i} D^i P_{\mathbf{w}}(N_x, U_x) - \sum_{x \in a'_i} D^i P_{\mathbf{w}}(N_x \cup \{i\}, U_x) \\
&= \sum_{x \in a_i} (P_{\mathbf{w}}(N_x, U_x) - P_{\mathbf{w}}(N_x \setminus \{i\}, U_x)) - \sum_{x \in a'_i} (P_{\mathbf{w}}(N_x \cup \{i\}, U_x) - P_{\mathbf{w}}(N_x, U_x)) \\
&= \sum_{x \in F} (P_{\mathbf{w}}(N_x, U_x) - P_{\mathbf{w}}(N'_x, U_x)) = \Phi(\mathbf{a}) - \Phi(a_{-i}, a'_i).
\end{aligned}$$

Thus Φ is a $\mathbf{w}$-potential function of G. $\square$

3.1 GCG + Shapley division are just congestion games

Since in congestion games the equal share of utility coincides with the Shapley value, Rosenthal's result that every congestion game has a potential function (the "easy" direction of the characterization) follows as a special case from Theorem 1. Since by Monderer and Shapley the other direction is also true, we get the following corollary:

Corollary 1. *Every GCG with Shapley division (as in the first part of Theorem 1), is isomorphic to a congestion game.*

In fact, given such a GCG G, we can construct an isomorphic (standard) congestion game $G^* = (N, F^*, (A_i^*)_{i\in N}, (U_x^*)_{x\in F})$ directly: For every resource $x \in F$, consider the unique decomposition of (N, U_x) to unanimity games, and in particular the coefficients $(\alpha_S)_{S\subseteq N}$. For every $x \in F$ we add $2^n - 1$ resources $\{x_S\}$ to F^*, one for each non-empty subset of N. We set $U^*_{x_S}(k) = \alpha_S$ if $k \geq |S|$, and $U^*_{x_S}(k) = 0$ otherwise. Although this is not required for our proof, we note that the value of the coefficient α_S can be written explicitly as

$$\alpha_S = U_x(S) - \sum_{T'\subseteq S, |T'|=|S|-1} U_x(T') + \sum_{T''\subseteq S, |T''|=|S|-2} U_x(T'') - \cdots \pm \sum_{i\in N} U_x(\{i\}),$$

i.e. using the Möbius transform [10]. Next, we define the strategy sets A_i^*. We map every $a_i \in A_i$ in the original GCG to a single strategy $a_i^* \subseteq F$ as $a_i^* = \{x_S \in F^* : x \in a_i, i \in S\}$. Then, we set $A_i^* = \{a_i^* : a_i \in A_i\}$.

Since $U^*_{x_S}$ is evenly divided among users of x_S, G^* is indeed a congestion game. It is left to prove that G^* is isomorphic to G. That is, that for every profile $\mathbf{a} = (a_j)_{j\in N}$ and every agent i, the utility of i in G is equal to the utility of i from profile $\mathbf{a}^* = (a_j^*)_{j\in N}$ in G^* (formally, that $u^{*i}(\mathbf{a}^*) = u^i(\mathbf{a})$).

First note that agent i selects resource x_S in $\mathbf{a}^*$ iff $i \in N_x$ *and* $i \in S$. Thus the resource x_S contributes non-zero utility only if the entire set S selects it (i.e. all $i \in S$ select x in $\mathbf{a}$).

For every $x \in F$, it holds that

$$\sum_{S\subseteq N_x} u^{*i}_{x_S}(N_{x_S}) = \sum_{S\subseteq N_x} u^{*i}_{x_S}(S) = \sum_{S\subseteq N_x : i\in S} \frac{U^*_{x_S}(S)}{|S|}$$
$$= \sum_{S\subseteq N_x : i\in S} \frac{\alpha_S}{|S|} \overset{\text{(Eq. (1))}}{=} \phi^i(N_x, U_x) = u^i_x(N_x).$$

That is, for each $x \in F$ the equal share of utility in G^* over all resources $\{x_S\}_{S\subseteq N_x}$, is equal to the utility gained from resource x in the original GCG. Finally, summing over all resources,

$$u^{*i}(\mathbf{a}^*) = \sum_{x_S\in F^*} u^{*i}_{x_S}(N_{x_S}) = \sum_{x\in F}\sum_{S\subseteq N_x} u^{*i}_{x_S}(N_{x_S}) = \sum_{x\in F} u^i_x(N_x) = u^i(\mathbf{a}).$$

4 Implications

It is not hard to see that any finite non-cooperative game can be represented as a very large GCG, with one resource for every profile of pure actions. Even if restricted to succinct representations, GCGs can still model an extremely wide class of games: any TU game can be used to construct each U_x. For example U_x can be based on weights of players [2], their skills [1], location [12], etc. As long as the total utility on each resource is distributed according to the Shapley value, the game will have a potential function.

4.1 Weighted variants of congestion games

In the two classes studied by Kollias and Roughgarden [5], costs are divided according to the Shapley value and the weighted Shapley value, respectively. Thus both follow as special cases from Theorem 1. Moreover, our theorem highlights the fact that although both classes were presented in the paper as variants of weighted congestion games (in the sense of [7]), the "weights" used in each result have different technical meaning. As said in the previous paragraph, *any* utility function would lead to a potential, and this includes functions that happen to be defined using weights. Further note that the first result of Kollias and Roughgarden can be substantially strengthened: utility does not have to be negative or monotone. Also, players may have different weights on each edge.

The role of weights in the application to cost-sharing games (the second result in [5]) is different. Here $U_x(N_x) = -\gamma_x$ is fixed and does not depend on weights at all. Thus dividing the cost according to the Shapley value simply means an equal share of the cost, as is the case in "standard" (unweighted) cost sharing games. True, such a division leads to a potential function but this is already known by Rosenthal [9]. The augmentation with weights $\boldsymbol{\beta}$ and dividing according to $\phi_{\boldsymbol{\beta}}$ allows for an additional β-weighted potential function to emerge.

To see why these implications are different, note that we could combine the two: take a weighted congestion game, where the total utility/cost is defined using weights $\boldsymbol{\beta}$ (i.e. $U_x(N_x) = U_x(\beta(N_x))$); now divide the utility according to some weighted Shapley value $\phi_{\mathbf{w}}$, then the game admits a $\mathbf{w}$-weighted potential function, regardless of $\boldsymbol{\beta}$.

4.2 Congestion games with failures

Following the notation of Meir et al. [6], We denote by $p(S : T)$ the probability that from the set T, exactly the agents of S survive . Formally, $p(S : T) = \sum_{R \subseteq N \setminus T} p(S \cup R)$. In particular, we denote by $p_i = p(\{i\} : \{i\})$ the marginal survival probability of i.

The cost of using resource x to each agent in the base game G is $c_x(|N_x|)$, for each agent that is using resource x. Applying the notation of GCG, the total utility in G is $U_x(N_x) = |N_x| u_x(N_x) = -|N_x| c_x(|N_x|)$, and it is equally shared between players of N_x. The perturbed game is denoted by $G^{\mathbf{p}}$. The cost of resource x to an agent $i \in N_x$ in the game $G^{\mathbf{p}}$ was defined by Meir et al. as the expected cost over all possible sets of survivors that include i. Formally,

$$c^{\mathbf{p}}_{i,x}(N_x) = \sum_{R \subset N_x \setminus i} p(R : N_x | i) c_x(|R| + 1) \quad (3)$$

$$= \sum_{R \subset N_x : i \in R} p(R : N_x | i) c_x(|R|). \quad (4)$$

As in standard congestion games, the total cost to player i (conditioned on survival) in profile $\mathbf{a}$ is $cost^{\mathbf{p}}_i(\mathbf{a}) = \sum_{x \in F} c^{\mathbf{p}}_{i,x}(N_x)$. If i does not survive, her utility is 0. Thus $u^{\mathbf{p}}_{i,x}(N_x) = -p_i c^{\mathbf{p}}_{i,x}(N_x)$, and $u^{\mathbf{p}}_i(\mathbf{a}) = p_i cost^{\mathbf{p}}_i(\mathbf{a})$.[1]

Note that $u^{\mathbf{p}}_{i,x}(N_x)$ may depend on the identity of i or of $N_x \setminus \{i\}$. Therefore, Meir et al. claim, $G^{\mathbf{p}}$ is not necessarily a congestion game. In any case it is easy to see that $G^{\mathbf{p}}$ is still a GCG, where

$$U^{\mathbf{p}}_x(N_x) = \sum_{S \subseteq N_x} p(S : N_x) U_x(S) = \sum_{i \in N_x} u^{\mathbf{p}}_{i,x}(N_x).$$

Potential functions For any fixed subset of survivors $T \subseteq N$, let Φ_T denote the potential function of the subgame $G|_T$ (which is a congestion game). The *weighted potential* of profile $\mathbf{a} = (N_x)_{x \in F}$ in the perturbed game $G^{\mathbf{p}}$ is defined as the convex combination of potential functions of all 2^n subgames. Formally,

$$\Phi^*(\mathbf{a}) = \sum_{T \subseteq N} p(T) \Phi_T(\mathbf{a}|_T) = \sum_{T \subseteq N} p(T) \sum_{x \in F} \sum_{k=1}^{|N_x \cap T|} c_x(k).$$

where $\mathbf{a}|_T = (N_x \cap T)_{x \in F}$. Meir et al. prove directly that for any profiles $\mathbf{a}, \mathbf{a}'$ that differ by the strategy of player i, $\Phi^*(\mathbf{a}) - \Phi^*(\mathbf{a}') = p_i(cost^{\mathbf{p}}_i(\mathbf{a}) - cost^{\mathbf{p}}_i(\mathbf{a}'))$. That is, that Φ^* is a weighted potential function.

We will next revisit this result in order to reach a sharper conclusion. By reversing the order of summation, we can write Φ^* as $\Phi^*(\mathbf{a}) = \sum_{x \in F} \Phi^*_x(N_x)$, where

$$\Phi^*_x(N_x) = \sum_{T \subseteq N_x} p(T : N_x) \sum_{k=1}^{|T|} c_x(k).$$

Let $D^i \Phi^*_x(N_x) = \Phi^*_x(N_x) - \Phi^*_x(N_x \setminus \{i\})$.

Lemma 2. $D^i \Phi_x(N_x) = p_i c^{\mathbf{p}}_{i,x}(N_x)$.

[1] In [6], p_i does not appear in the definition of $c^{\mathbf{p}}_{i,x}$, since the cost is conditioned on the fact that i survived.

Proof. $D^i\Phi^*_x(N_x) = \Phi^*_x(N_x) - \Phi^*_x(N_x \setminus \{i\})$

$$= \sum_{T\subseteq N_x} p(T:N_x)\sum_{k=1}^{|T|} c_x(k) - \sum_{T\subseteq N_x\setminus\{i\}} p(T:N_x\setminus\{i\})\sum_{k=1}^{|T|} c_x(k)$$

$$= \sum_{T\subseteq N_x\setminus\{i\}} [(1-p_i)p(T:N_x\setminus\{i\}|\neg i)\sum_{k=1}^{|T|} c_x(k) + p_i p(T:N_x|i)\sum_{k=1}^{|T|+1} c_x(k)]$$

$$- \sum_{T\subseteq N_x\setminus\{i\}} [(1-p_i)p(T:N_x\setminus\{i\}|\neg i)\sum_{k=1}^{|T|} c_x(k) + p_i p(T:N_x|i)\sum_{k=1}^{|T|} c_x(k)]$$

$$= p_i \sum_{T\subseteq N_x\setminus\{i\}} p(T:N_x|i)(\sum_{k=1}^{|T|+1} c_x(k) - \sum_{k=1}^{|T|} c_x(k))$$

$$= p_i \sum_{T\subseteq N_x\setminus\{i\}} p(T:N_x|i)c_x(|T|+1) = p_i c^{\mathbf{p}}_{i,x}(N_x). \qquad \square$$

Since $u^{\mathbf{p}}_{i,x}(N_x) = -p_i c^{\mathbf{p}}_{i,x}(N_x) = -D^i\Phi_x(N_x)$, the participation game on resource x (or, equivalently, the TU game induced by it) has an *exact* potential function Φ^*_x (the HMC potential). According to Lemma 1(a), this means that the total (negative) utility U_x in a game with failures, is necessarily divided according to the Shapley value of players using resource x.

Finally, by applying Theorem 1, we find that the entire game $G^{\mathbf{p}}$ has an exact (not weighted!) potential function, namely Φ^*. This means that every congestion game with failures is isomorphic to a "standard" congestion game - possibly with a much larger set of resources. It seems that Meir et al. [6] missed this point since they used the conditional cost $c^{\mathbf{p}}_i$ rather than the actual utility (not conditioned on survival of i).

5 Discussion

We defined a generalized class of congestion games, where any game in the class admits a [weighted] potential function as long as utility on each resource is divided according to the [weighted] Shapley value of agents using the resource. Several recent results from the literature follow as special cases, and can even be strengthened, as a consequence from this observation.

It would be interesting to find what other natural classes of games can be shown to admit a potential function using this approach.

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