Recall $\vec{r} : \mathbb{R} \to \mathbb{R}^3$, $\vec{r}(t) = (f(t), g(t), h(t))$

is a curve in space

$\vec{r}\,'(t) = (f'(t), g'(t), h'(t))$ is the

tangent vector to the curve $\vec{r}(t)$.

---

$\vec{r}(t) = t^2\hat{\imath} + (2t-1)\hat{\jmath} + t^3\hat{k}$, $t_0 = 2$

Find a parametric equation for the line tangent to the curve $\vec{r}(t)$ at $t = t_0$:

$\vec{r}(t_0) = \vec{r}(2) = (4, 3, 8) = 4\hat{\imath} + 3\hat{\jmath} + 8\hat{k}$

is the point at which we are looking for a tangent line

$\vec{r}\,'(t) = (2t, 2, 3t^2) = 2t\hat{\imath} + 2\hat{\jmath} + 3t^2\hat{k}$

$\vec{r}\,'(2) = (4, 2, 12) = 4\hat{\imath} + 2\hat{\jmath} + 12\hat{k}$

$\vec{l}(t) = \vec{r}(t_0) + \vec{r}\,'(t_0)t = (4,3,8) + (4,2,12)t$

13.2 Integrals of vector function : Projectile motion

$$\int \vec{r}(t)\, dt = \vec{R}(t) + \vec{C}$$

where $\vec{R}'(t) = \vec{r}(t)$ & $\vec{C}$ is a constant vector

$$\vec{C} = \underbrace{(c_1, c_2, c_3, \dots, c_n)}_{\text{Scalar constants}}$$

eg: $\vec{r}(t) = (t, t^2, \cos t)$

$\vec{R}(t) = (t^2/2, t^3/3, \sin t) + \vec{C}$

Integrate Component wise!

For definite Integral, same idea

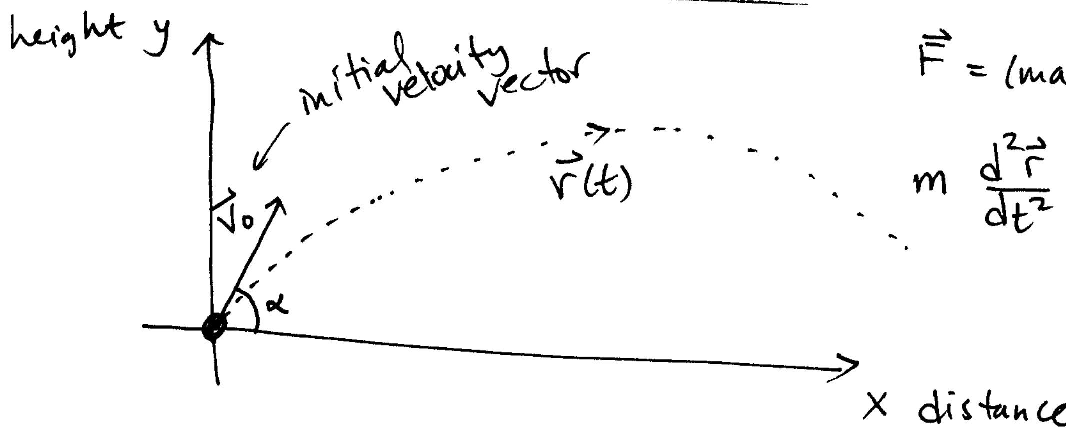
$$\int_a^b \vec{r}(t)\, dt = \vec{R}(t)\Big|_{t=a}^{t=b} \qquad \vec{R}(t) = (F, G, H)$$

$$= \left( F\Big|_a^b, G\Big|_a^b, H\Big|_a^b \right)$$

$$= \left( F(b) - F(a), G(b) - G(a), H(b) - H(a) \right)$$

Scalars (numbers)

# Ideal Projectile Motion Equation




$\vec{F}$ = (mass) x acceleration

$$m \frac{d^2\vec{r}}{dt^2} = -mg\hat{j}$$

$$\Rightarrow \frac{d^2\vec{r}}{dt^2} = -g\hat{j}$$

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt^2} = -g\hat{j} = (0, -g)$$

Integrate once to get velocity

$$\frac{d\vec{r}}{dt} = -gt\hat{j} + \vec{C}, \quad \frac{d\vec{r}}{dt} = \vec{v}(t) \text{ at } t=0$$

$$\left.\frac{d\vec{r}}{dt}\right|_{t=0} = \vec{v}(0) = \vec{v}_0 = \vec{C}$$

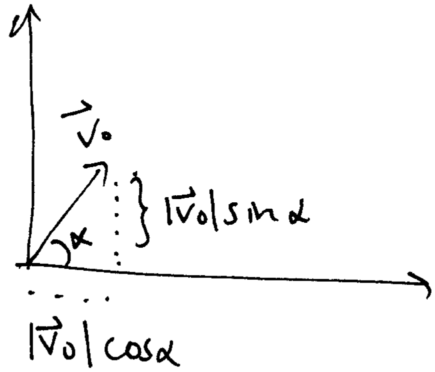
$$\vec{v}(t) = -gt\hat{j} + \vec{v}_0$$

Integrate again to get $\vec{r}(t)$ the position

$$\vec{r}(t) = -g\frac{t^2}{2}\hat{j} + \vec{v}_0 t + \vec{r}_0$$

In a simple setting as above assume $\vec{r}_0 = \vec{0}$ but otherwise follow what's given.

$$\vec{r}(t) = -g\frac{t^2}{2}\hat{j} + \vec{v}_0 t$$




$\alpha$ is initial launch angle

$$\vec{r}(t) = (|\vec{v}_0|\cos\alpha\, t)\,\hat{\imath} + \left(|\vec{v}_0|\sin\alpha\, t - \tfrac{1}{2}g t^2\right)\hat{\jmath}$$

$\alpha$ is the launch angle or angle of elevation or firing angle

$|\vec{v}_0|$ is the initial speed

Book uses $v_0 = |\vec{v}_0|$

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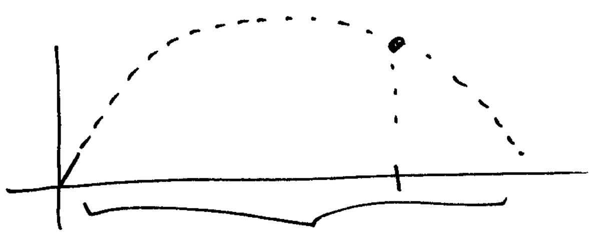
<u>Example</u>: A projectile is fired at 840 m/s at angle of 60°, How long will it take to get 21 km down range?

$g = 10\,m/s^2$
$.01$

Initial speed $|\vec{v}_0| = 840$ m/s $= .84$ km/s.
initial angle $\alpha = 60°$

$$\vec{r}(t) = .84\cos 60°\, t\,\hat{\imath} + \left(.84\, \cancel{\cos}\sin 60°\, t - \tfrac{1}{2}(.01\, t^2)\right)\hat{\jmath}$$



$\vec{r}(t) = (x(t), y(t))$

x distance travelled $x(t)$

Question asks for when is $x(t) = 21$ ?

$$.84 \cos 60^\circ\, t = 21 \Rightarrow t = 50 \text{ sec.}$$

$$\frac{.84}{2} t = 21 = t = \frac{21}{.42} = 50 \text{ sec.}$$

---

Ideal projectile motion $V_0$ = initial speed

$$x(t) = V_0 \cos\alpha\, t, \quad y(t) = V_0 \sin\alpha\, t - \frac{1}{2} g t^2$$

Solve for $t \Rightarrow \frac{x}{V_0 \cos\alpha} = t$

plug $t = \frac{x}{V_0 \cos\alpha}$ into $y(t)$

$$y = V_0 \sin\alpha \frac{x}{V_0 \cos\alpha} - \frac{1}{2} g \left(\frac{x}{V_0 \cos\alpha}\right)^2$$

$$y = \tan\alpha\, x - \frac{g}{2v_0^2 (\cos\alpha)^2} x^2$$

$$\cancel{y_{max} = \frac{(V_0 \sin\alpha)^2}{2g}} \quad y_{max} = \frac{(V_0 \sin\alpha)^2}{2g}$$

Flight time when $y = 0$

$$t = \frac{2v_0 \sin\alpha}{g}$$

$$\text{range} = \frac{V_0^2}{g} \sin 2\alpha.$$

$x$ value at $y = 0$

<u>Example:</u> In Moscow in 1987 Natalya Lisouskaya set a women's world record by putting an 8 lb 13 oz shot 73 ft 10 in. Assuming a 40° angle launch at 6.5 ft height what was the initial speed?

---

$$\vec{x}(t) = \vec{x}_0 + (v_0 \cos\alpha)t = 0 + v_0 \cos 40\, t \approx .766\, v_0 t$$

$$\vec{y}(t) = 6.5 + (v_0 \sin\alpha)t - \tfrac{1}{2} g t^2 = 6.5 + (v_0 \sin 40)t - 16t^2$$
$$\approx 6.5 + .643 v_0 t - 16t^2$$

Since shot went $73.83$ ft $\Rightarrow$ $X(\text{land time}) = 73.83$

$$t \approx \frac{96.383}{v_0} \text{ sec.}$$

Shot lands when $y(t) = 0 = 6.5 + .643(96.383) - 16\left(\frac{96.383}{v_0}\right)^2$

$$68.474 - \frac{148.63}{v_0^2} = 0 \Rightarrow v_0 \approx \sqrt{\frac{148}{68}} \approx 46.6 \text{ ft/sec}$$

How long did that take $t = \dfrac{96.3}{46.6}$ sec.

# 14.1 Functions of Several Variables

$f : \mathbb{R}^n \to \mathbb{R}$ is a scalar function of several variables ($n$)

eg: $f(x,y) = x + y$

$$f(x,y,z) = 3xyz - \frac{(\sin z)^2}{\ln(xyz)} + \cos^{-1}(z)$$

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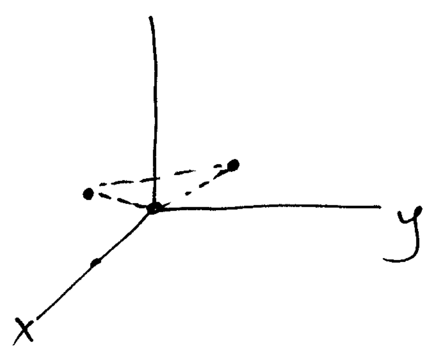
$f(x,y) = x+y$, let $\boxed{z = f(x,y)}$




$z = x + y$

$\Leftrightarrow z - x - y = 0$

$\boxed{-x - y + z = 0}$ ← Equation of plane $\vec{n} \cdot (x-0, y-0, z-0) = 0$

$\vec{n} = (-1, -1, 1)$