Sets and Mathematical Notation

A **set** X is a "well-defined" collection of objects. In this course, the objects will be (mostly) real numbers. To say that X is well-defined means that there must be an unambiguous test for determining whether or not something belongs to X. If x belongs to X, we write $x \in X$; if not, we write $x \notin X$. If $x \in X$, we say that x is an **element** of X.

Examples:

1. We can define a set by listing explicitly all the elements: $X = \{2, \pi, 3\}$.

2. We can define the elements of X by the requirement that they have a certain property: $X = \{x \mid x \text{ is a root of } x^3 + 5x^2 + 2\}$. In this context, the symbol $\mid$ is read as "such that".

3. It is not necessarily a trivial matter to determine whether or not $x \in X$. For instance, we can define $\mathbb{A} = \{x \mid x \text{ is a root of some polynomial with integer coefficients}\}$. Such numbers are called **algebraic numbers**. For instance, $\sqrt{2}$ is algebraic, since it's a root of $p(x) = x^2 - 2$. So is $e^{(2\pi i)/3}$, since it's a root of $p(x) = x^3 - 1$. It is true, but not easily proven, that neither π nor e is algebraic. Numbers that are not algebraic are called **transcendental**.

4. $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ is the set of **natural numbers**.

5. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is the set of **integers**. It consists of the natural numbers, their negatives, and the element 0.

6. $\mathbb{Q} = \{m/n \mid m, n \in \mathbb{Z},\ n \neq 0\}$ is the set of **rational numbers**.

7. $\mathbb{R}$ is the set of **real numbers**. The precise definition of $\mathbb{R}$ requires some care, and will be given later in the course. Similarly with $\mathbb{C}$, the set of **complex numbers**.

If X is a set, then A is a **subset** of X if every element of A belongs to X. We write this as $A \subseteq X$. For instance, $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. And for any set X, $X \subseteq X$. Two sets X and Y are said to be **equal** if each is a subset of the other: $X \subseteq Y$, and $Y \subseteq X$.

The **empty set**, denoted $\emptyset$, is the set containing no elements. For any set X, it is true that $\emptyset \subseteq X$, because every element of $\emptyset$ (there are none) belongs to X. It is also true that $X \subseteq X$ for any set X.

"Operations" on sets

- $X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$. This is not an "exclusive or"; x can belong to either X or Y, or both. $X \cup Y$ is called the **union** of the two sets X and Y.

- $X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$ is called the **intersection** of X and Y.

- $\sim X = \{x \mid x \notin X\}$ is called the **complement** of X. Here X is assumed to be a subset of some known set like the real numbers.
- $X \setminus A = \{x \mid x \in X \text{ and } x \notin A\}$ is called the **complement of A in X**. For instance, $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers.
- If X and Y are two sets, their **Cartesian product**, denoted $X \times Y$ is defined by

$$X \times Y = \{(x, y) \mid x \in X, \text{ and } y \in Y\}.$$

  That is, the Cartesian product of X and Y is the set of all ordered pairs, the first element of which belongs to X and the second to Y. If $X = Y$, then we write $X \times X$ as X^2. For instance, the plane $\mathbb{R}^2$ is the same as $\mathbb{R} \times \mathbb{R}$. Similarly, $X^n = X \times \cdots \times X$ (n times).

Quantifiers

- The symbol $\forall$ is read "for all". Equivalently, "for every", "for any" and "for each". For instance

$$A \subseteq X \text{ means } \forall x \in A, x \in X.$$

- The symbol $\exists$ is read "there exists". Equivalently, "for some", "there is", or "one can find". For instance,

$$A \text{ is not a subset of } X \text{ if } \exists x \in A \setminus X. \ (A \setminus X \neq \emptyset)$$

Implications

A **proposition** (also, depending on the context, called a theorem, lemma, or corollary) is a (grammatically correct) sentence of the form "if A, then B". This is written $A \Rightarrow B$, and is read as "A implies B". That is, if A is true, then B must be true. A is said to be "sufficient" for B, and B is "necessary" for A. Propositions are also called **implications**. A well-formed proposition is either true or false, and formal mathematics can be said to consist of the proof or disproof of mathematical propositions. (Actually, there is a lot of tricky business involved with set theory, and it's a famous theorem of Godel that there exist well-formed statements whose truth or falsity cannot be determined. But we'll ignore this minor issue, since it doesn't arise here.)

To **prove** the implication $A \Rightarrow B$, you assume that A is true, and then attempt to show that B is also true. To **disprove** it, you must find an example for which A holds, but B does not.

Examples:

1. $0 < x < 1 \Rightarrow x^2 < x$. Here, A is the statement "Suppose x is any real number in $(0, 1)$.", and $\Rightarrow$ is read "then", while B is the statement "$x^2 < x$." Proof: Assume that x is a real number in $(0, 1)$. Then, since $x > 0$, multiplying both sides of an inequality by x doesn't reverse the inequality. Since $x < 1$, we have $x \cdot x = x^2 < x \cdot 1 = x$.

2. The **converse** of $A \Rightarrow B$ is the implication $B \Rightarrow A$. The converse of $A \Rightarrow B$ may be either true or false, regardless of the truth or falsity of $A \Rightarrow B$. The converse of the implication above is the implication $x^2 < x \Rightarrow 0 < x < 1$. In this case, the converse is true. Here is an example where the converse is false: $x^2 = 5 \Rightarrow x \notin \mathbb{Q}$.

3. The **contrapositive** of the implication $A \Rightarrow B$ is the implication $\neg B \Rightarrow \neg A$, where the symbol $\neg$ is read as "not". The contrapositive is logically equivalent to the original implication. For instance, "If Amy is human then Amy is a mammal." The contrapositive is "If Amy is not a mammal, then Amy is not human". We often use the contrapositive implication to prove something - this is called a "proof by contradiction". The contrapositive of (1) is the statement $x^2 \geq x \Rightarrow x \notin (0,1)$. Proof: Suppose that $x^2 \geq x$. If $x \in (0,1)$, then we can show (above) $x^2 < x$. Therefore, $x \notin (0,1)$.

Double implications

An implication of the form $A \iff B$ is really two implications, $A \Rightarrow B$, and $B \Rightarrow A$. That is, both $A \Rightarrow B$ and its converse are true. Both must be proven to establish the truth of this statement. In English, we read $A \iff B$ as "A implies and is implied by B", or "A is true if and only if (iff) B is true" or (as our forefathers might say, "A is necessary and sufficient for B").

Definitions should be read as double implications: for instance

$$A \subseteq B \iff \forall x \in A,\ x \in B.$$

That is, if $A \subseteq B$, then any element of A belongs to B. And conversely, if any element of A belongs to B then $A \subseteq B$. If you're given that $A \subseteq B$, you know that any element of A belongs to B. On the other hand, if A and B are two sets and you can show that any element of A belongs to B, then you've proven that $A \subseteq B$.

Another implied double implication is involved in proving the equality of two sets. For example, to show

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$$

you must show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and conversely. To do this, you start by assuming that $x \in A \cap (B \cup C)$ and show that $x \in (A \cap B) \cup (A \cap C)$. Then do the converse.

Negations

Negations can be a bit tricky. Obviously, we replace "is" by "is not". We also replace $\exists$ with $\forall$ and vice versa. For example, the contrapositive of the implication

$$\forall x \in A,\ x \in B \Rightarrow A \subseteq B$$

is the statement

$$\neg(A \subseteq B) \Rightarrow \neg(\forall x \in A,\ x \in B),\ \text{or}$$

$$A \text{ is not a subset of } B \Rightarrow \exists x \in A \mid x \notin B.$$

In English, the right hand side of the implication reads "for some $x \in A$, x is not in B".