Equivalence relations

DEFINITION: Let X be a set. A **relation** on X is a subset R of the product $X \times X$. If $(x, y) \in R$, then we say that "x is related to y".

You're already familiar with one example of a relation:

DEFINITION: $R \subset X \times X$ is a **function** if $(x, y),\ (x, z) \in R \Rightarrow y = z$.

EXAMPLES:

- $X = \mathbb{R}, R = \{(x, y) : y = x^2\}$. This is a function, since if $(x, y) \in R$ then $y = x^2$, and if $(x, z) \in R,\ z = x^2$, and so $y = z$. In the case of a function $f : X \to X$, the relation is simply the **graph** of f.
- $X = \mathbb{R}, R = \{(x, y) : x = y^2\}$ is not a function.

NOTATION: For equivalence relations (next definition), we use a special notation and write "x is related to y" as $x \sim y$.

DEFINITION: An **equivalence relation** $\sim$ on the set $X \times X$ is a relation that satisfies the following three conditions:

1. It is **reflexive**, which means that $x \sim x,\ \forall x \in X$.
2. It is **symmetric**, which means that if $x \sim y$, then $y \sim x$. (Note that arbitrary x, y are not necessarily related, but if x is related to y, then the symmetry implies that y is related to x.
3. It is **transitive**, which means that $x \sim y$ and $y \sim z \Rightarrow x \sim z$.

Even though relations are defined on the product of two sets, we usually say, somewhat inaccurately, that "$\sim$ is an equivalence relation on X".

Examples: (Of course, you should check that each of these satisfies the three conditions).

1. $X = \mathbb{R}$ with the relation $x \sim y \iff x = y$. This is the "original" equivalence relation; the more general concept arose from trying to abstract some of the most useful properties of equality. Any sort of equality, such as congruence of triangles, is generally an equivalence relation.
2. $X =$ the set of all triangles in the plane. For $x, y \in X,$ define $x \sim y \iff x$ and y are similar.
3. Let $X =$ the set of all $n \times n$ matrices with real entries. If $A, B \in X$, define $A \sim B \iff \exists P$, an $n \times n$ *non-singular* matrix, such that $B = P^{-1}AP$.

4. Let $X = \mathbb{Z}$ and define $m \equiv n (\text{mod } 4) \iff m - n$ is divisible by 4. That is, m and n have the same remainder when divided by 4. This is a equivalence relation on $\mathbb{Z}$ known as **congruence mod** 4.

5. $X = \mathbb{Z} \times \{\mathbb{Z} \setminus \{0\}\} = \{(m, n) : m, n \in \mathbb{Z},\ n \neq 0\}$. Define $(m, n) \sim (s, t) \iff mt = ns$. It is easily checked that this is an equivalence relation.

Now suppose $\sim$ is an equivalence relation on X. We define the **equivalence class** of $x \in sX$, denoted $[x]$:

$$[x] = \{y \in X \mid y \sim x\}.$$

Thus, in example (2), $[x]$ is the set of all triangles similar to x.

As for example (3), recall from your linear algebra course that if a linear transformation T is represented in some basis by the matrix A, and P changes over to a new basis, then B represents the same linear transformation T in the new basis. Thus $[A]$ is the set of all possible matrix representations of the linear transformation T.

In example (4), we can just list the equivalence classes: They are $\{[0], [1], [2], [3]\}$. Remember, $[1]$ is a *set*, so for instance, $[1] = [5] = [9] = \cdots$.

In example (5), the equivalence classes can be identified with the rational numbers $\mathbb{Q}$.

PROPOSITION: IF $\sim$ IS AN EQUIVALENCE RELATION ON X, AND $[x], [y]$ ARE TWO EQUIVALENCE CLASSES, THEN EITHER $[x] = [y]$ OR $[x] \cap [y] = \emptyset$.

PROOF: (exercise)

So an equivalence relation partitions the original set X into a collection of disjoint (i.e., non-intersecting) subsets (the equivalence classes). In some sense, the relation of equality on X is a "trivial" equivalence relation - trivial in the sense that it's not very interesting, since the equivalence classes are just the original members of the set X. What is the subset R of $X \times X$ corresponding to the relation of equality?

The other trivial equivalence relation is gotten by taking $R = X \times X$ - every pair of points is related, and there's just one equivalence class: for any $x \in X$, $[x] = X$. The interesting equivalence relations lie between these two extremes.

Operations on equivalence classes: Sometimes, an operation like addition defined on the original set X can also be defined on the equivalence classes. For instance in example 4, we have the four equivalence classes

$$[0] = \{\dots, -8, -4, 0, 4, 8, 12, \dots\}$$

$$[1] = \{\dots, -7, -3, 1, 5, 9, 13, \dots\}$$

$$[2] = \{\dots, -6, -2, 2, 6, 10, 14, \dots\}$$

$$[3] = \{\dots, -5, -1, 3, 7, 11, 15, \dots\}$$

Every integer is in one and only one of these classes.

Pick any element in [3]. It has the form of $3 + 4n$ for some integer n. For instance, we can take $3 - 2 \cdot 4 = -5$. Similarly, any element of the class [1] is of the form $1 + 4m$ for some integer m. We can take $1 + 4 \cdot 5 = 21$. If we add the two representatives, we get $-5 + 21 = 16 = 4 \cdot 4$, which belongs to the equivalence class [0], since it's a multiple of 4. In fact, for any n, m we get

$$(3 + 4n) + (1 + 4m) = 4 + 4(n + m) = 0 + 4(n + m + 1) \equiv 0 (\text{mod } 4).$$

So it makes sense to define $[3] + [1] = [0]$, and so on. This is called addition mod 4. You already know several examples:

- Angles about the origin in $\mathbb{R}^2$ are only defined up to the arbitrary addition of integral multiples of 2π. We write, for instance, $\pi/3 \equiv -5\pi/3 (\text{mod } 2\pi)$ (since $\pi/3 - (-5\pi/3) = 2\pi$). From this point of view, the circle is the real line "rolled up" in such a way that the point x is identified with any point of the form $x + 2k\pi, k \in \mathbb{Z}$.
- If it's now 10:00 o'clock, then in 7 hours it will be 5:00 o'clock. This is arithmetic "(mod 12)". (Because $10 + 7 = 17 \equiv 5 (\text{mod } 12)$.) Military time uses arithmetic (mod 24).

Two other examples of interest here are (a) the construction of $\mathbb{Q}$ from $\mathbb{Z}$, indicated above, and (b) the construction of $\mathbb{R}$ from the set of all Cauchy sequences in $\mathbb{Q}$ (later in the course).