Cardinal numbers

☐ Definition: A set X is **finite** if for some $n \in \mathbb{N}$ there's a bijection $f : X \to \{1, 2, \ldots, n\}$. We say that the **cardinal number** of X is n and write $\mathrm{Card}(X) = n$.

☐ Definition: A set X is **infinite** if it's not finite. (For any natural number n, no bijection between X and $\{1, 2, \ldots, n\}$ exists.) Examples of infinite sets include $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$, and $\mathbb{R}$, (which we haven't yet defined!)

☐ Definition: A set X is **countably infinite** if there's a bijection $f : X \to \mathbb{N}$. For example, the set $\mathbb{Z}$ is countable: define

$$f(n) = \begin{cases} 0 & \text{if } n = 0 \\ 2n & \text{if } n > 0 \\ -2n - 1 & \text{if } n < 0 \end{cases}$$

☐ Definition: A set is **countable** if it's finite or countably infinite - that is, if there's a bijection from X onto a subset (possibly all) of $\mathbb{N}$.

☐ Definition: A countably infinite set is said to have cardinality $\aleph_0$. (Read "aleph nought" - this is the Hebrew letter aleph (a).) So $\mathrm{Card}(\mathbb{N}) = \mathrm{Card}(\mathbb{Z}) = \aleph_0$.

Countably infinite sets have some strange properties. Suppose you have $\mathbb{N}$ hotel rooms, and they're all booked. You can accommodate a new guest by moving the person in room 1 to room 2, ..., room n to room n+1, etc., and putting the new guest in room 1. If a countably infinite number of new guests appear, you can move the current occupant of room n to room 2n, and put the new guests into rooms 1, 3, 5, ...

A set is countable if you can produce an algorithm which makes an ordered list out of the elements of the set, and exhausts the set. That is, if you have a method for selecting a first element, then a second one, etc., and you can show that, proceeding in this fashion, you eventually list every element of the set, then you can write x_1 for the first element, x_2 for the second and so on. The bijection is $f(n) = x_n$. For example, with the set $\mathbb{Z}$, our algorithm is to write the elements in the order $0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots$. The function defined above corresponds to this listing.

Proposition: The union of two countable sets is countable. From this it follows that the union of any finite number of countable sets is countable. In fact, the union of a countable number of countable sets is countable.

Proof: Suppose $X_1, X_2, \ldots, X_n, \ldots$ is a countable collection of countable sets. Denote the elements of X_1 by

$$X_1 = \{x_{11}, x_{12}, x_{13}, x_{14}, \ldots\},$$

and the elements of X_n by

$$X_n = \{x_{n1}, x_{n2}, x_{n3}, x_{n4}, \ldots\}.$$

Now make a 2-dimensional listing of all the elements of these sets. It looks like

$$\begin{array}{c} x_{11}, x_{12}, x_{13}, x_{14}, \ldots \\ x_{21}, x_{22}, x_{23}, x_{24}, \ldots \\ x_{31}, x_{32}, x_{33}, x_{34}, \ldots \\ x_{41}, x_{42}, x_{43}, x_{44} \ldots \\ \vdots \end{array}$$

Then starting from the upper left corner of this array, you work your way through the entire list by going up the diagonals. The listing is

$$x_{11}, x_{21}, x_{12}, x_{31}, x_{22}, x_{13}, x_{41}, x_{32}, x_{23}, x_{14}, x_{51}, \ldots$$

Can you write out the function $f(n)$?

PROPOSITION: IF X AND Y ARE COUNTABLE, THEN SO IS $X \times Y$.

PROOF: This is not much different from the preceding proof. Since X is countable, we can list its elements: $X = \{x_1, x_2, x_3, \ldots\}$, and similarly for Y. The Cartesian product is then $X \times Y = \{(x_i, y_j) \mid i, j \in \mathbb{N}\}$. Then use the same proof as above, identifying the ordered pair (x_i, y_j) with x_{ij} in the list above.

♣ Exercise: Prove the following:

1. Any subset of a countable set is countable.
2. $\mathbb{Q}$ is countable.

□ Definition: A set X which is not countable is said to be (you guessed it) **uncountable**. Since such a set is not countable, it's not finite. Nor can it be put in 1-1 correspondence with $\mathbb{N}$.

Assuming, for the moment, that we know what real numbers are, there's a cute proof, due originally to Cantor, that $\mathbb{R}$ is uncountable. What we'll show is that the open unit interval is uncountable, using the decimal (or binary, or hexadecimal, etc.) representations of real numbers:

Suppose, to the contrary, that the set of real numbers in $(0, 1)$ is countable. Then, as we've seen, we can write them out in an ordered list: $x_1, x_2, x_3, \ldots$. Now each real number x_n has a decimal expansion which we write as

$$x_n = .a_{n1}a_{n2}a_{n3}a_{n4}\cdots$$

where each a_{nm} is in $\{0, 1, \ldots, 9\}$. Now consider the following decimal expansion,

$$b = .b_1b_2b_3b_4\cdots,$$

where each integer b_n is chosen so that $b_n \neq a_{nn}$. (There are 9 possible choices for each b_n; it doesn't matter which choice is made.) So $b \neq x_n$ for any n since the decimal expansion of b differs from that of x_n in the n^{th} slot. Therefore b cannot be in the list of the x_ns. But clearly $b \in (0, 1)$, so this is a contradiction - the supposed list did not contain all the numbers in the interval. Since the set $(0, 1)$ is uncountable, and is a subset of $\mathbb{R}$, it follows that $\mathbb{R}$ is uncountable as well.