### Uniform convergence and its consequences

The following issue is central in mathematics: On some domain D, we have a sequence of functions  $\{f_n\}$ . This means that we really have an uncountable set of "ordinary" sequences, since for each  $x \in D$ , we have the sequence  $\{f_n(x)\}$ .

### **1** Pointwise convergence

 $\Box$  Definition: Suppose that a sequence of functions  $f_n : D \to \mathbb{R}$  is given and that, for each  $x \in D$ ,

$$\lim_{n \to \infty} f_n(x) \text{ exists.}$$

We write f(x) for this limit, and say that the sequence of functions  $\{f_n\}$  converges pointwise to f on the domain D.

EXAMPLE 1: On  $(0, \infty)$ , let

$$f_n(x) = \frac{nx - 1}{nx}.$$

Then

$$\lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \frac{1 - 1/(nx)}{1} = 1, \forall x > 0,$$

so  $f_n$  converges pointwise to the constant function f(x) = 1.

EXAMPLE 2: On (-1, 1), let

$$f_n(x) = 1 + x + x^2 + \dots + x^n.$$

Then, as we know,  $f_n$  converges pointwise to the limit 1/(1-x). Of course, in this case, we write the limit as the sum of an infinite series:

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{ for } |x| < 1.$$

The issue at hand is the behavior of the limit function f. For instance, suppose each of the functions  $f_n$  is continuous; is it true that f is continuous? More generally, which properties of the sequence  $\{f_n\}$  survive the limiting process? As another example, suppose that

$$\sum_{n=0}^{\infty} a_n x^n$$
 converges to some  $f(x)$  on  $D$ .

Here the sequence of functions is the sequence of partial sums of the series:

$$f_n(x) = \sum_{k=0}^n a_k x^k.$$

Each member  $f_n$  of the sequence is a polynomial; it can be differentiated, for instance:

$$f'_{n}(x) = \sum_{k=1}^{n} k a_{k} x^{k-1}.$$

What can we say about the sequence  $\{f'_n\}$ . Does it converge? If so, does it converge to f', the derivative of the original limit? For the geometric series above, pointwise convergence of the sequence of derivatives is equivalent to the convergence of the infinite series

$$\sum_{n=1}^{\infty} nx^{n-1},$$

which can be demonstrated using the ratio test:

$$\left|\frac{b_{n+1}}{b_n}\right| = \left(\frac{n+1}{n}\right)|x| \to |x| \text{ as } n \to \infty,$$

so the sequence of derivatives of the geometric series converges on the same interval as the original series. But what's the limit? We suspect that

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2},$$

but if true, this must be shown separately - it doesn't follow from the above.

Naturally, things do not always work out the way we imagine. Here are some important counterexamples to bear in mind:

- 1.  $f_n(x) = x^n$  on the interval [0, 1]. Here, for  $0 \le x < 1$ , we have  $x^n \to 0$ . On the other hand, for  $x = 1, x^n \to 1$ . So here is a case of a sequence of continuous functions converging pointwise on a closed interval to a limit function which is *not* continuous.
- 2. Define  $g_n : [0,1] \to \mathbb{R}$  by

$$g_n = \begin{cases} 0 & : \quad x = 0\\ n & : \quad 0 < x \le \frac{1}{n}\\ 0 & : \quad \frac{1}{n} < x \le 1 \end{cases}$$

For each  $x \in (0, 1]$ , there exists an  $n \in \mathbb{N}$  such that  $\frac{1}{n} < x$ . And therefore  $g_n(x) \to 0$  as  $n \to \infty$ . And for x = 0,  $g_n(0) = 0$ ,  $\forall n$ . We therefore have pointwise convergence of the sequence  $g_n(x)$  to the limit function g(x) = 0. On the other hand, we have

$$\int_{0}^{1} g_{n}(x)dx = 1, \forall n, \text{ while } \int_{0}^{1} g(x)dx = 0.$$

Here the sequence of functions converges pointwise to a continuous limit, but the integrals of the functions do not converge to the integral of the limit. The specific functions  $g_n$  are not continuous, but that's not the source of the problem: instead of step functions, we could use "tent" functions which are continuous, and achieve the same unpleasant result.

- **&** Exercise: (1) Construct a sequence of *continuous* functions on [0, 1] which converges pointwise to the the zero function and whose integrals behave as above.
- 3. Let  $f_n(x) = x^n/n$  on the interval [0, 1]. This sequence converges pointwise everywhere to the function f(x) = 0. But the sequence  $f'_n(x)$  is the same as the sequence in the first example. So here is an example of a sequence of functions converging to a continuous limit, but the derivatives of the sequence do not converge pointwise to the derivative of the limit (there's a problem at x = 1, as above).
- 4. Let

$$s_n(x) = \sum_{k=1}^n (-1)^{k+1} \frac{\sin kx}{k}.$$

We have argued (not rigorously) that  $s_n(x) \to f(x) = x/2$ , and in particular, that  $s_n(\pi/2) \to \pi/4$ . Over what interval are these considerations valid? What about the sequence of derivatives, or the sequence of integrals?

## 2 Uniform convergence

The concept we need to make sense of this is called *uniform convergence*.

 $\Box$  Definition: The sequence  $f_n$  of functions defined on the domain D is said to converge uniformly to the function f on D if for any  $\epsilon > 0$ ,  $\exists n \in \mathbb{N}$  such that for all  $x \in D$ ,  $|f_n(x) - f(x)| < \epsilon$ .

This means that (a) every one of the sequences  $\{f_n(x)\}$  converges to f(x), and that (b) the number n in the definition works for every  $x \in D$ . In some sense, every single one of these sequences converges at the same rate.

Geometrically, if  $f_n \to f$  uniformly on [a, b], and  $\epsilon > 0$  is given, draw a ribbon extending a distance  $\epsilon$  in either direction from the graph of f on [a, b]. Then there exists an N such that the graph of  $f_n$  lies inside the ribbon for all  $n \ge N$ .

 $\Box$  Definition: The infinite series  $\sum a_n(x)$  is said to be uniformly convergent in D if the sequence of partial sums is a uniformly convergent sequence of functions in D.

EXAMPLES: (1) The sequence  $f_n(x) = x^n$  converges, but the convergence is not uniform: suppose that  $0 < \epsilon < 1$ , 0 < x < 1. Then  $|f_n(x) - f(x)| < \epsilon \iff x^n < \epsilon \iff x < \epsilon^{1/n}$ . But since  $\epsilon^{1/n} < 1$ , for any *n* we can always find an  $x \in (0, 1)$  for which the inequality is violated. So the convergence is not uniform. (2) On the other hand, for  $f_n(x) = x^n/n$  the convergence to the limit function f(x) = 0 is uniform, since

$$\left|\frac{x^n}{n}\right| \le \frac{1}{n} < \epsilon \iff n > \frac{1}{\epsilon}.$$

**&** Exercise: (2) Show that the sequence of functions you constructed in exercise 1 above is not uniformly convergent.

**&** Exercise: (3) Let  $f_n(x) = \sum_{k=0}^n x^k$ , and f(x) = 1/(1-x). If 0 < r < 1, show that  $f_n \to f$  uniformly in the interval [-r, r].

### **3** Consequences of uniform convergence

One of the main tools used to demonstrate uniform convergence is called the *Weierstrass* M-test; it's a comparison test:

THEOREM: Let the terms of the series  $\sum a_n(x)$  be defined on some domain D, and let  $\sum M_n$  be a convergent series of positive constants. If the inequality  $|a_n(x)| \leq M_n$  holds for all n and for all  $x \in D$ , then the series  $\sum a_n(x)$  is uniformly convergent on D.

PROOF: We will show that Cauchy's convergence criterion is satisfied: if  $s_n(x)$  is the  $n^{th}$  partial sum of the series, then, if m < n,

$$s_n(x) - s_m(x) = a_{m+1}(x) + \dots + a_n(x)$$
, and so  
 $|s_n(x) - s_m(x)| \le M_{m+1} + \dots + M_n$ ,

since  $|a_k(x)| \leq M_k$ . Since the series  $\sum M_n$  is convergent, if  $\epsilon > 0$  is given, there exists  $N \in \mathbb{N}$  such that if  $N \leq m < n$ ,  $M_{m+1} + \cdots + M_n < \epsilon$ , and so, independent of x, we have  $|s_n(x) - s_m(x)| < \epsilon$ , which establishes the uniform convergence of  $\sum a_n(x)$  in D.

EXAMPLE: The series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

is uniformly convergent on  $\mathbb{R}$ . Here we may take as a comparison series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}, \text{ since } |\sin nx| \le 1,$$

and the series  $\sum (1/n^2)$  is known to converge.

THEOREM: If the functions  $f_n$  are continuous in D, and  $f_n \to f$  uniformly in D, then f is continuous.

PROOF: Let  $\epsilon > 0$  be given. We need to show that, for any  $x_0 \in D$ , we can find a  $\delta > 0$  such that  $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$ . We have

$$|f(x) - f(x_0)| = |f(x) - f_n(x) + f_n(x) - f_n(x_0) + f_n(x_0) - f(x_0)|$$
  

$$\leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$$

Since  $f_n$  converges uniformly to f, we can find an N such that  $n \ge N \Rightarrow |f_n(x) - f(x)| < \epsilon/3$  for all  $x \in D$ . From the last inequality, we then have

$$|f(x) - f(x_0)| \le |f_n(x) - f_n(x_0)| + 2\epsilon/3, \ \forall n \ge N.$$

Now fix any such n; since  $f_n$  is continuous, there's a  $\delta > 0$  such that  $|x - x_0| < \delta \Rightarrow |f_n(x) - f_n(x_0)| < \epsilon/3$ . So  $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$ , which is what we had to show.

COROLLARY: If the terms of an infinite series are continuous on D and if the series converges uniformly on D, then the sum of the series is a continuous function.

# 4 Integration and differentiation of uniformly convergent sequences

We need to use some simple facts about integration here. In subsequent classes, we'll prove the existence of the Riemann integral for continuous functions on closed intervals; here we'll just assume it.

We'll need the following

**PROPOSITION:** If  $f : [a, b] \to \mathbb{R}$  is continuous, then so is |f|, and

$$\left| \int_{a}^{b} f(x) dx \right| \le \int_{a}^{b} |f(x)| dx$$

**PROOF** : The inequality holds for all Riemann sums. Hence it holds in the limit for the integral.

THEOREM: Suppose the functions  $f_n$  are continuous on [a, b] and converge uniformly to f on [a, b]. Then

$$\int_{a}^{b} f(x)dx = \lim_{n \to \infty} \int_{a}^{b} f_{n}(x)dx$$

(For an infinite series, this is equivalent to the statement that if the series  $\sum a_n(x) = f(x)$  is uniformly convergent on [a, b], and if each  $a_n(x)$  is continuous on [a, b], then

$$\int_{a}^{b} f(x)dx = \sum_{n} \int_{a}^{b} a_{n}(x)dx.$$

To make sure you understand this, write out exactly why this second statement is true.) PROOF: We need to show that, given an  $\epsilon > 0$ , we can make the quantity

$$\left|\int_{a}^{b} f_{n}(x)dx - \int_{a}^{b} f(x)dx\right|$$

less than  $\epsilon$  for *n* sufficiently large. By the proposition above,

$$\left|\int_{a}^{b} f_{n}(x)dx - \int_{a}^{b} f(x)dx\right| \leq \int_{a}^{b} |f_{n}(x) - f(x)|dx.$$

And since the sequence is uniformly convergent, there exists an  $n_0$  such that

$$n \ge n_0 \Rightarrow |f_n(x) - f(x)| < \frac{\epsilon}{b-a}, \ \forall x \in [a, b].$$

But then

$$\int_{a}^{b} |f_{n}(x) - f(x)| dx < \int_{a}^{b} \frac{\epsilon}{b-a} dx = \epsilon,$$

so we're done.

EXAMPLE: We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \text{ if } |x| < 1.$$

So choose any r such that 0 < r < 1. From the theorem, we know that

$$\int_{-r}^{r} \frac{1}{1-x} dx = \sum_{n=0}^{\infty} \int_{-r}^{r} x^n dx$$
$$= \sum_{n=0}^{\infty} \int_{-r}^{r} x^{2n} dx \text{ (n odd gives 0)}$$

Evaluating the integrals gives

$$\log \frac{1+r}{1-r} = 2\left(r + \frac{r^3}{3} + \frac{r^5}{5} + \cdots\right)$$

THEOREM: Suppose  $f(x) = \lim_{n\to\infty} f_n(x)$ , where the convergence is uniform on [a, b]. Suppose also that the derivatives  $f'_n(x)$  are continuous and converge uniformly on [a, b]. Then f(x) is differentiable, and

$$f'(x) = \lim_{n \to \infty} f'_n(x).$$

**PROOF:** Since  $f'_n$  is a uniformly convergent sequence of continuous functions, we may write

$$g(x) = \lim_{n \to \infty} f'_n(x),$$

where the limit function g is continuous on [a, b]. We need to show that g(x) = f'(x). Using the previous result on term-by-term integration, we may write, for any  $x \in [a, b]$ ,

$$\int_{a}^{x} g(t)dt = \lim_{n \to \infty} \int_{a}^{x} f'_{n}(t)dt.$$

But then

$$\int_{a}^{x} g(t)dt = \lim_{n \to \infty} \left[ f_n(x) - f_n(a) \right]$$
$$= f(x) - f(a),$$

where in the second line, we've used the uniform convergence of the original sequence  $f_n$ . Taking the derivatives of both sides, we have g(x) = f'(x), and the theorem is proven.

**&** Exercise: COROLLARY: If the series  $\sum a_n(x)$  converges uniformly to f(x) on [a, b], and if the derivatives  $a'_n(x)$  are continuous and also converge uniformly on the interval, then

$$f'(x) = \sum_{n=1}^{\infty} a'_n(x),$$

so the series may be differentiated term-by-term.

EXAMPLES: Consider the following expressions:

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + \dots$$
$$\frac{2}{(1-x)^3} = \sum_{n=2}^{\infty} n(n-1)x^{n-2} = 2 + 6x + 12x^2 + \dots$$

These are obtained by differentiating both sides of the expression for the geometric series

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Are the two expressions valid? The answer is positive if it can be shown that the series on the right hand sides converges uniformly. If so, then the corollary asserts the equality of the two sides. We consider the first series. We know that the original series converges for |x| < 1. Choose any number r such that 0 < r < 1. Then we have, for all  $x \leq r$ ,

$$|kx^{k-1}| \le kr^{k-1} =: M_k.$$

But the series

$$\sum_{k=1}^{\infty} kr^{k-1} = \sum_{k=1}^{\infty} M_k$$

converges by the ratio test since

$$\lim_{k \to \infty} \frac{M_{k+1}}{M_k} = \lim_{k \to \infty} \frac{k+1}{k}r = r < 1.$$

So, by the Weierstrass M-test, the series of derivatives converges uniformly for |x| < r. Since r is an arbitrary positive number less than 1, the series of derivatives converges uniformly on the interval |x| < 1, and the first expression is valid.

**&** Exercise: Show that the second expression is valid as well.