Uniform Continuity

Recall that if f is continuous at x_0 in its domain, then for any $\epsilon > 0$, $\exists \delta > 0$ such that for all x in the domain of f, $|x - x_0| < \delta \Longrightarrow |f(x) - f(x_0)| < \epsilon$. The number δ will generally depend on x_0. In the future, we shall omit the phrase "for all x in the domain", which is to be understood.

- $f(x) = x^2$ is continuous at any $x_0 \in \mathbb{R}$.

  Proof: Let $\epsilon > 0$ be given, and suppose that $|x - x_0| < \delta$. (We haven't fixed δ yet, we're just fooling around.). Then

  $$|f(x) - f(x_0)| = |x^2 - x_0^2| = |x - x_0||x + x_0|;$$

  and $|x - x_0| < \delta \Longrightarrow$

  $$\begin{array}{ccccc} -\delta + x_0 & < & x & < & -\delta + x_0 \\ 2x_0 - \delta & < & x + x_0 & < & 2x_0 + \delta. \end{array}$$

  And, since $-2|x_0| - \delta < 2x_0 - \delta$, and $2x_0 + \delta < 2|x_0| + \delta$, we have $|x + x_0| < 2|x_0| + \delta$, so that

  $$|f(x) - f(x_0)| = |x - x_0||x + x_0| < \delta[2|x_0| + \delta].$$

  Since x_0 is fixed, the right hand side of this inequality can be made as small as we like by choosing δ sufficiently small, and so we can find a δ such that the right hand side is $< \epsilon$.

  For instance, if $x_0 = 2$, we require $\delta(4 + \delta) < \epsilon$. And if $x_0 = 476$, we need $\delta(952 + \delta)$. It should be clear that as $x_0 \to \infty$, $\delta \to 0$, and this means that there is no single δ which works for every $x_0 \in \mathbb{R}$.

- $f(x) = 1/x$ on $[1/2, 1]$. Let $\epsilon > 0$ be given, and let $x, y \in [1/2, 0]$ with $|x - y| < \delta$. Then

  $$|f(x) - f(y)| = \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|x - y|}{|xy|}.$$

  Since $x, y \in [1/2.1]$, $|xy| \geq (\frac{1}{2})^2 = \frac{1}{4} \Longrightarrow \dfrac{1}{|xy|} \leq 4$. So if $|x - x_0| < \delta$,

  $$|f(x) - f(y)| = \frac{|x - y|}{|xy|} < 4\delta < \epsilon \text{ if } \delta < \epsilon/4.$$

Here, δ does *not* depend on x and y. It works for *any* pair of numbers in the interval $[1/2, 1]$ with $|x - y| < \delta$.

Definition: A function $f : D \to \mathbb{R}$ is said to be **uniformly continuous** on $D \iff \forall \epsilon > 0$, $\exists \delta > 0$ such that $|x - y| < \delta \Longrightarrow f(x) - f(y)| < \epsilon$.

The function $f(x) = x^2$ is continuous, but not uniformly continuous on $\mathbb{R}$, as we've seen above. The function $f(x) = 1/x$ is uniformly continuous on the closed interval $[1/2, 1]$. It's also continuous on the open interval $(0, 1)$, but we might expect problems with uniform continuity since this function is unbounded on its domain (as is $f(x) = x^2$):

- The function $f(x) = 1/x$ is not uniformly continuous on $(0, 1)$:

  PROOF: To show this, we need to find a "counterexample"; namely, we need to show

  $$\exists \epsilon_0 > 0 \text{ such that } \forall\ \delta > 0,\ \exists\ x, y \in (0,1) \text{ with } |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \epsilon_0.$$

  With a little thought, this is easily done. Take $\epsilon_0 = 1$ and choose any $\delta > 0$ (we may assume $\delta < 1$, since if our example holds for this case, it certainly holds for $\delta \geq 1$. Then put $x = \delta/3,\ y = \delta/6,$ so $|x - y| = \delta/6 < \delta$. And now

  $$|f(x) - f(y)| = \frac{|x - y|}{|xy|} = \frac{\delta/6}{\delta^2/18} = \frac{3}{\delta} \geq 1 = \epsilon_0.$$

We are going to need the notion of uniform continuity to prove the existence of the Riemann integral for continuous functions in a bit. The main fact we need is given by the following

THEOREM: LET $f : [a, b] \to \mathbb{R}$ BE CONTINUOUS ON THE CLOSED INTERVAL $[a, b]$. THEN f IS UNIFORMLY CONTINUOUS ON $[a, b]$.

PROOF: (This is a mildly intricate proof, but it doesn't involve anything new or surprising. After you've understood it, you should be able to write it out yourself.)

Suppose f is continuous but not uniformly continuous. Then there exists an $\epsilon_0 > 0$ such that the statement "$|x - y| < \delta \implies |f(x) - f(y)| < \epsilon_0$" is false for every δ. In particular, for $\delta = 1/n$, we can find two points $x_n, y_n \in [a, b]$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$. This gives us two sequences $\{x_n\}$ and $\{y_n\}$, both lying in $[a, b]$, and with $|x_n - y_n| \to 0$.

REMARK: Notice that *if* $x_n \to x$ and $y_n \to y$, then we'd have $x = y$, and because f is continuous, $|f(x_n) - f(y_n)| \to 0$, which would be a contradiction (why?). Now there's no reason to believe $x_n \to x$ or $y_n \to y$, but since these sequences lie in the closed interval $[a, b]$, there are convergent subsequences, and this is how we'll obtain the contradiction.

Continuing with the proof, by the Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, and a point $x \in [a, b]$, such that $x_{n_k} \to x$. (This is where we're using the fact that the inteval is closed.) Now put $\tilde{x}_k = x_{n_k}$, and $\tilde{y}_k = y_{n_k}$. So $\tilde{x}_k \to x$, and $\tilde{y}_k$ is a subsequence of the original sequence y_n. By the same reasoning, there exists a $y \in [a, b]$ and a subsequence $\{y_{k_s}\}$ with $y_{k_s} \to y$. Put $\hat{x}_s = \tilde{x}_{k_s},\ \hat{y}_s = \tilde{y}_{k_s}$. Then $\hat{x}_s$ is a subsequence of a convergent subsequence and hence also converges to x. Since $\hat{y}_s \to y$, and we have $|\hat{x}_s - \hat{y}_s| \to 0$, we must have $|f(\hat{x}_s) - f(\hat{y}_s)| \to 0$ (why?). This contradicts the assumption that $|f(x_n) - f(y_n)|$ is bounded away from 0 (why?). Therefore, our assumption that f is not uniformly continuous is incorrect.

Exercises:

1. Why, in the proof of the theorem, can't we just take a convergent subsequence of x_n and a convergent subsequence of y_n and proceed directly to the conclusion?
2. If f is differentiable on $[a, b]$, then f is uniformly continuous on $[a, b]$.
3. Show that $\sin x$ is uniformly continuous on $\mathbb{R}$. (See the proof of the differentiability of $\sin x$.