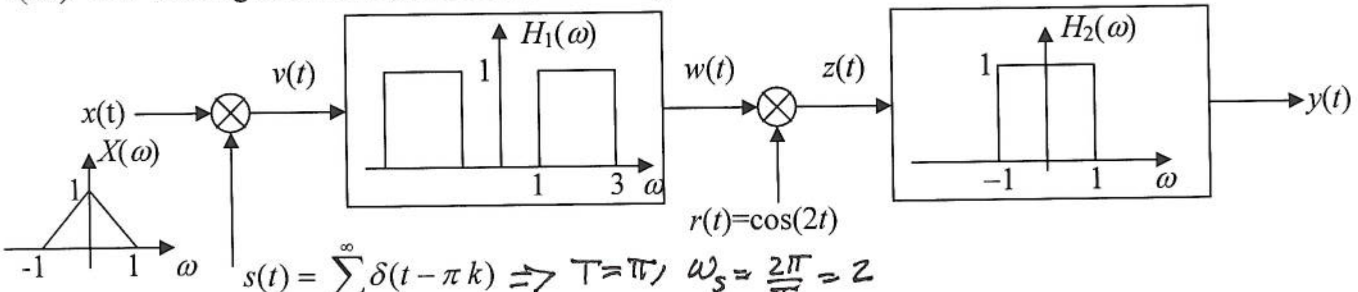
Assignment #13 - Solutions

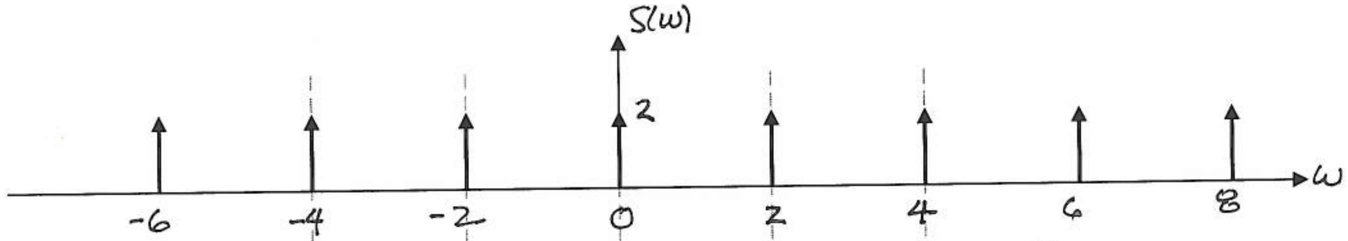
ECSE-2410 Signals & Systems - Spring 2013 Due Fri 10/25/13

1(35). For the diagram shown, sketch the following waveforms. Label all key values.

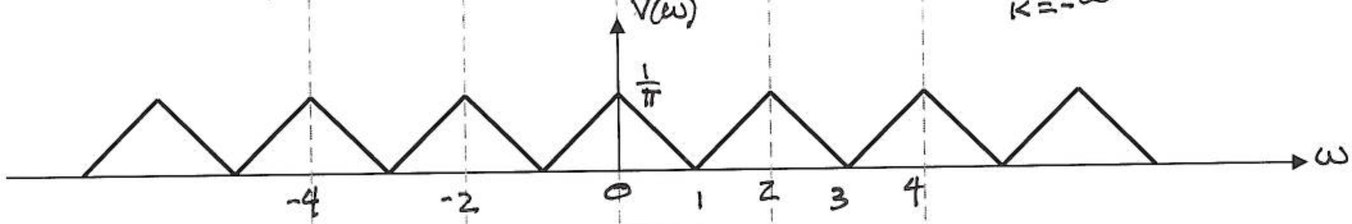


$s(t)=\sum_{k=-\infty}^{\infty}\delta(t-\pi k) \Rightarrow T=\pi,\ \omega_s=\frac{2\pi}{T}=2$

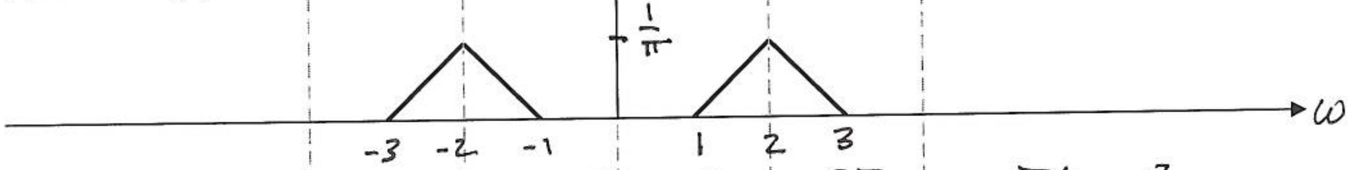
(a) Sketch $S(\omega)$. $s(t)=\sum_{k=-\infty}^{\infty}\frac{1}{\pi}e^{jk\omega_0 t} \leftrightarrow S(\omega)=\sum_{k=-\infty}^{\infty}\frac{1}{\pi}2\pi\delta(\omega-k\omega_s)=\sum_{k=-\infty}^{\infty}2\,\delta(\omega-2k)$



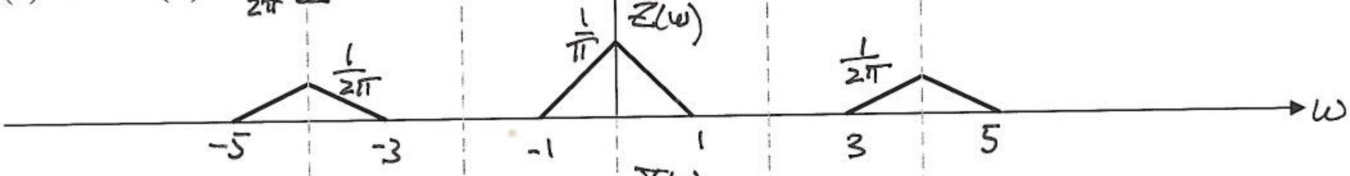
(b) Sketch $V(\omega)$. $=\frac{1}{2\pi}X(\omega)*S(\omega)=\frac{1}{2\pi}X(\omega)*\sum 2\delta(\omega-2k)=\frac{1}{\pi}\sum_{k=-\infty}^{\infty}X(\omega-2k)$



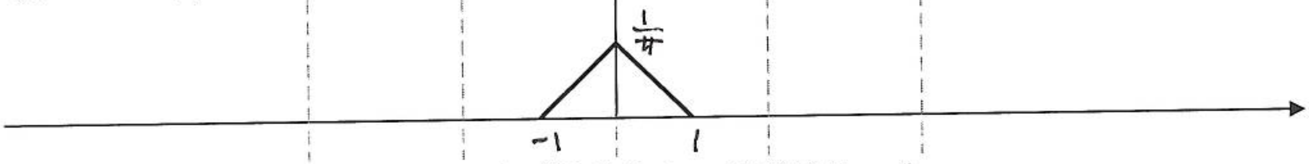
(c) Sketch $W(\omega)$.



(d) Sketch $Z(\omega)$. $=\frac{1}{2\pi}W(\omega)*\pi[\delta(\omega-2)+\delta(\omega+2)]=\frac{1}{2}[W(\omega-2)+W(\omega+2)]$

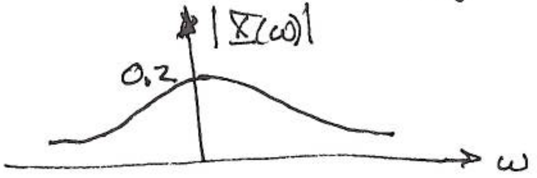


(e) Sketch $Y(\omega)$.



2(20). Are the following signals bandlimited? Explain why or why not.

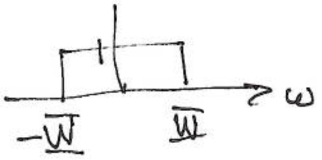
(a) $x(t)=e^{-5t}u(t)$ $\longleftrightarrow$ $\overline{X}(\omega) = \frac{1}{5+j\omega}$



$\boxed{\text{Not bandlimited}}$

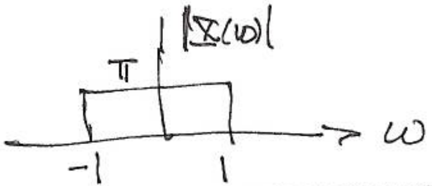
(b) $x(t)=\text{sinc}(t)$

Using $\frac{\overline{W}}{\pi}\,\text{sinc}(\overline{W}t) \longleftrightarrow$



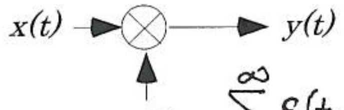
$\overline{W} = 1$

$\text{sinc}(t) \longleftrightarrow$



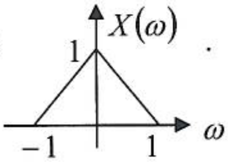
$\boxed{\text{yes, Bandlimited with } \omega_{max} = 1}$

3(20). Two signals are multiplied as shown, where $c(t)$ is the impulse train, $c(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$, with $T = \frac{2\pi}{3}$ sec.



$$c(t) = \sum_{K=-\infty}^{\infty} \delta(t-kT) \leftrightarrow C(\omega) = \frac{2\pi}{T} \sum_{K=-\infty}^{\infty} \delta(\omega - K\omega_s), \; \omega_s = \frac{2\pi}{T} = \frac{2\pi}{\frac{2\pi}{3}} = 3$$

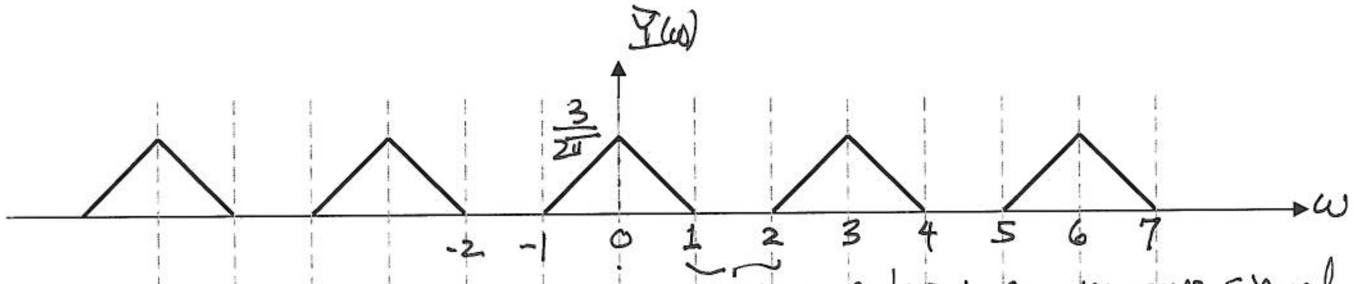
(a) Let the spectrum of $x(t)$ be



. Sketch the spectrum of $y(t)$, $Y(\omega)$. Can $x(t)$ be recovered from $y(t)$ using lowpass filtering? Why or why not?

$$y(t) = x(t)\,c(t) \leftrightarrow Y(\omega) = \frac{1}{2\pi} X(\omega) * C(\omega) = \frac{1}{2\pi} X(\omega) * \sum_{K=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - K\omega_s) = \frac{1}{T} \sum_{K=-\infty}^{\infty} X(\omega - K\omega_s)$$

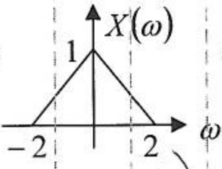
For $T = \frac{2\pi}{3}$, $\omega_s = 3$ & $Y(\omega) = \frac{3}{2\pi} \sum_{K=-\infty}^{\infty} X(\omega - K3)$



No overlap ∴ can recover signal

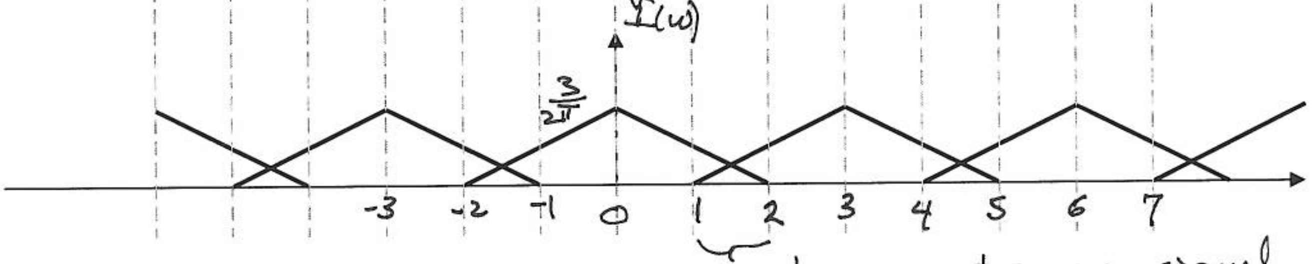
since $\omega_s = 3 > 2\omega_{max} = 2$

(b) Repeat part (a) if the spectrum of $x(t)$ is



and $c(t)$ remains the same. Can $x(t)$ be recovered from $y(t)$ using lowpass filtering? Why or why not?

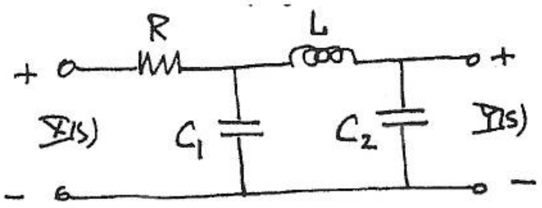
$\omega_{max} = 2$



overlap ∴ cannot recover signal

Since $\omega_s = 3 \not> 2\omega_{max} = 4.$

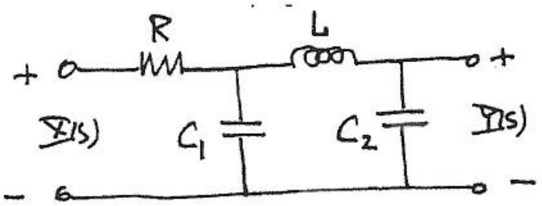
4(25). Find the transfer function $H(s) = \dfrac{Y(s)}{X(s)}$, where $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{2}$, $L = 4$, $R = 3$.



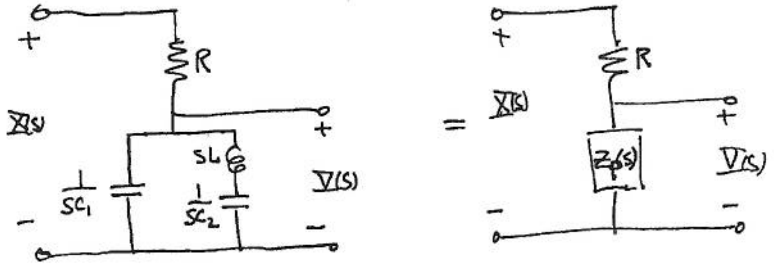
From: **Note #43. Butterworth Filters**

3. Realization of Filter as RCL Circuit.

Verify that the third order Butterworth filter transfer function, $B(s) = \dfrac{\omega_c^3}{s^3 + 2\omega_c s^2 + 2\omega_c^2 s + \omega_c^3}$, is realized by the circuit. Express the elements C_1, C_2, L in terms of R and the cutoff frequency, ω_c.



Method 1. First find the circuit transfer function using the voltage divider law. This is done in two steps. First find $V(s)$ and then $Y(s)$.



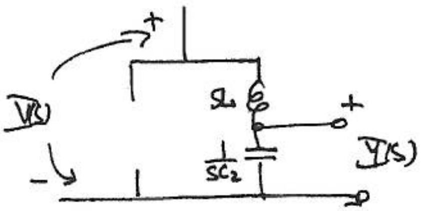
Therefore,

$$\frac{V(s)}{X(s)} = \frac{Z_p(s)}{R + Z_p(s)}, \text{ where } Z_p(s) = \frac{\left(\frac{1}{sC_1}\right)\left(\frac{1}{sC_2} + sL\right)}{\frac{1}{sC_1} + \frac{1}{sC_2} + sL} = \frac{1 + s^2 LC_2}{s(C_1 + C_2) + s^3 LC_1C_2},$$

and

$$\frac{V(s)}{X(s)} = \frac{1 + s^2 LC_2}{sR(C_1 + C_2) + Rs^3 LC_1C_2 + 1 + s^2 LC_2}.$$

Expressing $Y(s)$ in terms of $V(s)$



results in

$$\frac{Y(s)}{V(s)}=\frac{\frac{1}{sC_2}}{sL+\frac{1}{sC_2}}=\frac{1}{1+s^2LC_2}$$

Combining the two transfer functions,

$$\frac{Y(s)}{X(s)}=\frac{V(s)}{X(s)}\cdot\frac{Y(s)}{V(s)}=\frac{1+s^2LC_2}{sR(C_1+C_2)+Rs^3LC_1C_2+1+s^2LC_2}\cdot\frac{1}{1+s^2LC_2}$$

$$=\frac{\frac{1}{RLC_1C_2}}{s^3+s^2\left(\frac{1}{RC_1}\right)+s\left(\frac{C_1+C_2}{LC_1C_2}\right)+\frac{1}{RLC_1C_2}}$$

Comparing this equation term by term to

$$B(s)=\frac{\omega_c^3}{s^3+2\omega_c s^2+2\omega_c^2 s+\omega_c^3}$$

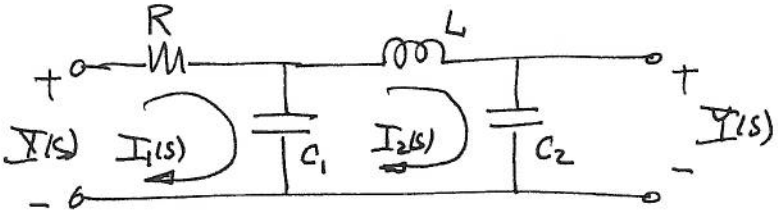
yields

$$\frac{1}{RLC_1C_2}=\omega_c^3,\ \ \frac{1}{RC_1}=2\omega_c,\ \ \frac{C_1+C_2}{LC_1C_2}=2\omega_c^2$$

Solving simultaneously for L, C_1, C_2, gives us

$$C_1=\frac{1}{2R\omega_c},\ \ C_2=\frac{3}{2R\omega_c},\ \ L=\frac{4R}{3\omega_c}$$

Method #2. Using Kirchhoff's voltage law



results in the equations

$$\left(R+\frac{1}{sC_1}\right)I_1(s)-\frac{1}{sC_1}I_2(s)=X(s)$$
$$-\frac{1}{sC_1}I_1(s)+\left(\frac{1}{sC_1}+sL+\frac{1}{sC_2}\right)I_2(s)=0 .$$

Simplifying,

$$(RsC_1+1)I_1(s)-I_2(s)=sC_1X(s)$$
$$-sC_2I_1(s)+\left(s^3LC_1C_2+s(C_1+C_2)\right)I_2(s)=0 .$$

Now solve for $I_2(s)$ using Cramer's rule,

$$I_2(s)=\frac{\begin{vmatrix} RsC_1+1 & sC_1X(s) \\ -sC_2 & 0 \end{vmatrix}}{\begin{vmatrix} RsC_1+1 & -1 \\ -sC_2 & s^3LC_1C_2+s(C_1+C_2) \end{vmatrix}},$$

or

$$I_2(s)=\frac{sC_2X(s)}{s^3LRC_1C_2+s^2LC_2+sR(C_1+C_2)+1}.$$

Then the output is

$$Y(s)=\frac{1}{sC_2}I_2(s)=\frac{X(s)}{s^3LRC_1C_2+s^2LC_2+sR(C_1+C_2)+1}$$

resulting in the transfer function,

$$H(s)=\frac{Y(s)}{X(s)}=\frac{1}{s^3LRC_1C_2+s^2LC_2+sR(C_1+C_2)+1}$$

or in standard form,

$$H(s) = \frac{\frac{1}{LRC_1C_2}}{s^3 + s^2\left(\frac{1}{RC_1}\right) + s\left(\frac{C_1 + C_2}{LC_1C_2}\right) + \frac{1}{LRC_1C_2}}.$$

For $C_1 = \frac{1}{6}$, $C_2 = \frac{1}{2}$, $L = 4$, $R = 3$

$$\frac{1}{LRC_1C_2} = \frac{1}{4(3)\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)} = 1$$

$$\frac{1}{RC_1} = \frac{1}{3\left(\frac{1}{6}\right)} = 2$$

$$\frac{C_1 + C_2}{LC_1C_2} = \frac{\frac{1}{6} + \frac{1}{2}}{4\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)} = \frac{\frac{8}{12}}{\frac{4}{12}} = 2$$

$$\therefore H(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

A third order Butterworth filter.