香 港 中 文 大 學
The Chinese University of Hong Kong

二0一三至一四年度下學期科考試

Course Examination 2nd Term, 2013-14

科目編號及名稱
Course Code & Title : ENGG2430A Enginneering Mathematics III

時間
Time allowed : 2 小時 hours 00 分鐘 minutes

學號
Student I.D. No. : ____________ 座號 Seat No. : ____________

Instructions:

- Write your complete student ID on the answer book AND any additional paper.
- The exam is open-book and open-note. The total time allowed is 2 hours. There are 4 questions altogether. The full mark is 100.
- You must explain your answer clearly to receive full credit.
- Some statistical tables are available at the end of this question paper.
- You are allowed to use calculators approved by CUHK for examinations. However, you are not allowed to use your mobile phones, computers or any other electronic devices.

Part A

1. (15 points) There are 100 soldiers. One of them has a special disease, and the rest 99 are healthy. We select a solider uniformly at random, and put him through a medical test. The test is 95% accurate. That is, if the soldier has the disease, then the test result will be positive with probability 0.95 and be negative with probability 0.05. Similarly, if the solider is healthy, then the test result will be negative with probability 0.95 and be positive with probability 0.05.

   The randomly-selected soldier performs the test and the result is positive.

   (a) (5 points) Your friend Tom says that since the test result is positive and the test is 95% accurate, thus the probability that the soldier has the decease is 0.95. Tom's argument is wrong because he confuses with two conditional probabilities. Can you help Tom by pointing out the two probabilities?

   Solution: Let $A = \{\text{The solider has the disease}\}$, $B = \{\text{the test result is positive}\}$. The probability 0.95 given by the problem setting is the conditional probability $\Pr(B|A)$. Tom confuses it with a different conditional probability $\Pr(A|B)$.

   (b) (10 points) Given the test result is positive, what is the probability that the randomly-selected soldier has the disease?

   Solution: By Total Probability Law and Bayes' Rule, we can compute

$$\Pr(A|B) = \frac{\Pr(B|A)\Pr(A)}{\Pr(B)} = \frac{0.95 \times 0.01}{0.01 \times 0.95 + 0.99 \times 0.05} \approx 0.16$$

2. (15 points) Researchers perform a study of the monthly income of fresh engineering graduates from University A.

   (a) The mean $\bar{\mathsf{X}}$ of a random sample of size $n = 36$ is used to estimate the mean μ of the monthly income of fresh engineering graduates from University A that has standard deviation $\sigma = 4$ thousand dollars. What can we assert about the probability that the error ($|\bar{\mathsf{X}} - \mu|$) will be less than 1 thousand dollars, if we use

      i. (4 points) Chebyshev's theorem;

      Solution: Given $n = 36$,$\sigma = 4$ and $\frac{k\sigma}{\sqrt{n}} = 1$, we have $k = \frac{3}{2}$. Thus,

$$\Pr\left(|\bar{\mathsf{X}} - \mu| < \frac{k\sigma}{\sqrt{n}} = 1\right) \geq 1 - \frac{1}{(3/2)^2} = \frac{5}{9} \approx 0.5556$$

      ii. (6 points) the central limit theorem?

Solution: Given $n = 36$ and $\sigma = 4$, we have

$$\begin{aligned}
\Pr\left(\left|\bar{\mathsf{X}} - \mu\right| < 1\right) &= \Pr\left(-1 < \bar{\mathsf{X}} - \mu < 1\right) \\
&= \Pr\left(\frac{-1}{4/\sqrt{36}} < \bar{\mathsf{X}} - \mu < \frac{1}{4/\sqrt{36}}\right) \\
&= \Pr(-1.5 < \mathsf{Z} < 1.5) \qquad \text{where } \mathsf{Z} \sim N(0,1) \\
&= \Phi(1.5) - \Phi(-1.5) \\
&\approx 0.9332 - 0.0668 \\
&= 0.8664
\end{aligned}$$

(b) (5 points) Suppose that the mean and standard deviation of the monthly income are, respectively, $\mu = 20$ thousand dollars and $\sigma = 4$ thousand dollars. A random sample of size $n = 64$ fresh engineering graduates are selected. Find the probability that the average monthly income of those graduates is more than 21 thousand dollars.

Solution: Given $\mu = 20$, $\sigma = 4$, $n = 64$, we have

$$\begin{aligned}
\Pr\left(\bar{\mathsf{X}} > 21\right) &= \Pr\left(\frac{\bar{\mathsf{X}} - \mu}{\sigma/\sqrt{n}} > \frac{21 - 20}{4/\sqrt{64}}\right) \\
&= \Pr(\mathsf{Z} > 2) \qquad \text{where } \mathsf{Z} \sim N(0,1) \\
&= 1 - \Phi(2) \\
&\approx 1 - 0.9772 \\
&= 0.0228
\end{aligned}$$

Part B

3. (35 points) Suppose the scores of the final exam are normally distributed. You want to find out the mean and the variance from a random sample of 16 students. The sample mean and standard deviation are 62 points and 5 points respectively.

(a) (10 points) Give a 95% confidence interval of the population mean.

Solution: The sample size is $n = 16$, the observed sample mean and standard deviation are $\bar{x} = 62$ and $s = 5$ respectively. Since $n < 30$ and the population is normal, the sample mean has a t-distribution with $n - 1 = 15$ degrees of freedom. With $\alpha = 0.05$, the desired $(1 - \alpha)$ confidence interval is

$$\begin{aligned}\bar{x} \pm t_{\alpha/2,n-1}\frac{s}{\sqrt{n}} &= 62 \pm t_{0.05/2,15}\frac{5}{\sqrt{16}} \\ &\approx 62 \pm 2.131\frac{5}{4} = 62 \pm 2.6637 \approx [59.3, 64.7]\end{aligned}$$

(b) (10 points) Give a 95% confidence interval of the population variance.

Solution: With $\alpha = 0.05$, the desired $(1 - \alpha)$ confidence interval is

$$\begin{aligned}\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right] &\approx \left[\frac{(16-1)5^2}{\chi_{0.05/2}}, \frac{(16-1)5^2}{\chi_{1-0.05/2}}\right] \\ &\approx \left[\frac{(15)5^2}{27.49}, \frac{(15)5^2}{6.262}\right] \\ &\approx [13.6, 59.9]\end{aligned}$$

(c) (15 points) Suppose your score is 60 points and you believe you are above the population mean. Under a significance level of 5%, does the sample suggest that you should reject your belief? Give the test statistics, null hypothesis, alternative hypothesis, critical region and the final decision of the hypothesis test.

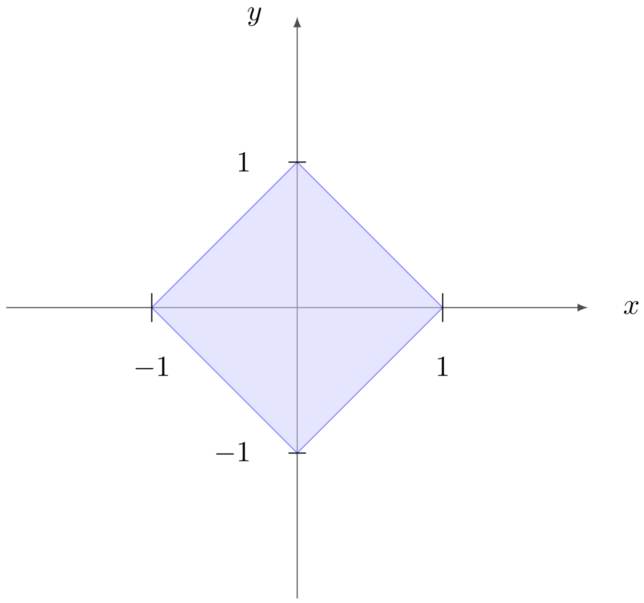
Solution: From the question, the null and alternative hypotheses are:

$$\begin{aligned}H_0 &: \mu = \mu_0 \\ H_1 &: \mu > \mu_0\end{aligned}$$

where μ denotes the population mean and $\mu_0 = 60$.
The critical region is "$t > t_{\alpha,n-1} \approx 1.753$", where t denotes the t-value computed from the data. In particular, $t := \frac{\bar{x}-\mu_0}{s/\sqrt{n}} = \frac{62-60}{5/4} \approx 1.6$, which does not lie in the critical region. Therefore, we do *not* reject the null hypothesis.

4. (35 points) A point (X, Y) is uniformly randomly chosen from the shaded region defined below.

$$\mathcal{R} := \{(x, y) : x + y \in [-1, 1], x - y \in [-1, 1]\}$$



(a) (10 points) Give the joint pdf of X and Y.

Solution: The area of $\mathcal{R}$ is $(\sqrt{1^2+1^2})^2 = 2$, and so the joint pdf is

$$f_{\mathsf{XY}}(x,y) = \begin{cases} \frac{1}{2} & (x,y) \in \mathcal{R} \\ 0 & \text{otherwise} \end{cases}.$$

(b) (15 points) Compute the expectations and variances of X and Y.

Hint: You may save some calculations using the following fact. If we define $\mathsf{U} := \mathsf{X} + \mathsf{Y}$ *and* $\mathsf{V} := \mathsf{X} - \mathsf{Y}$*, then* U *and* V *are independent and uniformly distributed over* $[-1, 1]$.

Solution: With U and V as defined in the hint, we can rewrite (X,Y) in terms of (U,V), as

$$\mathsf{X} := \frac{\mathsf{U}+\mathsf{V}}{2} \qquad \text{and} \qquad \mathsf{Y} := \frac{\mathsf{U}-\mathsf{V}}{2}$$

By the linearity of expectation,

$$E[\mathsf{X}] = \frac{E[\mathsf{U}] + E[\mathsf{V}]}{2} = 0 = \frac{E[\mathsf{U}] - E[\mathsf{V}]}{2} = E[\mathsf{Y}]$$

because $E[\mathsf{U}] = E[\mathsf{V}] = 0$. By the independence between U and V,

$$\mathrm{Var}(\mathsf{X}) = \mathrm{Var}(\mathsf{Y}) = \frac{\mathrm{Var}(\mathsf{U}) + \mathrm{Var}(\mathsf{V})}{2^2} = \frac{\mathrm{Var}(\mathsf{U})}{2} = \frac{1}{6}$$

because $\mathrm{Var}(\mathsf{U}) = E[\mathsf{U}^2] = \int_{-1}^{1} u^2 \frac{1}{2}\, du = \frac{1}{3}$.

(c) (10 points) Are X and Y independent? If not, compute the covariance of X and Y.

Solution: Note that Y is uniformly distributed over $[-1, 1]$ given $\mathsf{X} = 0$. However, Y

must be 0 given $\mathsf{X} = 1$. Therefore X and Y are *not* independent. The covariance is

$$\begin{aligned}
\mathrm{Cov}(\mathsf{X}, \mathsf{Y}) &= \mathrm{Cov}\left(\frac{\mathsf{U}+\mathsf{V}}{2}, \frac{\mathsf{U}-\mathsf{V}}{2}\right) \\
&= \frac{1}{2^2}\,\mathrm{Cov}(\mathsf{U}+\mathsf{V}, \mathsf{U}-\mathsf{V}) \\
&\overset{(a)}{=} \frac{1}{2^2}\left[\mathrm{Var}(\mathsf{U}) - \mathrm{Var}(\mathsf{V})\right] \overset{(b)}{=} 0
\end{aligned}$$

where (a) is by independence and (b) is because U and V are identically distributed.

Table 1: CDF $\Phi(z)$ of Standard Normal Distribution

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-2	0.02275	0.0233	0.02385	0.02442	0.025	0.02559	0.02619	0.0268	0.02743	0.02807
-1.9	0.02872	0.02938	0.03005	0.03074	0.03144	0.03216	0.03288	0.03362	0.03438	0.03515
-1.8	0.03593	0.03673	0.03754	0.03836	0.0392	0.04006	0.04093	0.04182	0.04272	0.04363
-1.7	0.04457	0.04551	0.04648	0.04746	0.04846	0.04947	0.0505	0.05155	0.05262	0.0537
-1.6	0.0548	0.05592	0.05705	0.05821	0.05938	0.06057	0.06178	0.06301	0.06426	0.06552
-1.5	0.06681	0.06811	0.06944	0.07078	0.07215	0.07353	0.07493	0.07636	0.0778	0.07927
-1.4	0.08076	0.08226	0.08379	0.08534	0.08691	0.08851	0.09012	0.09176	0.09342	0.0951
-1.3	0.0968	0.09853	0.1003	0.102	0.1038	0.1056	0.1075	0.1093	0.1112	0.1131
-1.2	0.1151	0.117	0.119	0.121	0.123	0.1251	0.1271	0.1292	0.1314	0.1335
-1.1	0.1357	0.1379	0.1401	0.1423	0.1446	0.1469	0.1492	0.1515	0.1539	0.1562
-1	0.1587	0.1611	0.1635	0.166	0.1685	0.1711	0.1736	0.1762	0.1788	0.1814
-0.9	0.1841	0.1867	0.1894	0.1922	0.1949	0.1977	0.2005	0.2033	0.2061	0.209
-0.8	0.2119	0.2148	0.2177	0.2206	0.2236	0.2266	0.2296	0.2327	0.2358	0.2389
-0.7	0.242	0.2451	0.2483	0.2514	0.2546	0.2578	0.2611	0.2643	0.2676	0.2709
-0.6	0.2743	0.2776	0.281	0.2843	0.2877	0.2912	0.2946	0.2981	0.3015	0.305
-0.5	0.3085	0.3121	0.3156	0.3192	0.3228	0.3264	0.33	0.3336	0.3372	0.3409
-0.4	0.3446	0.3483	0.352	0.3557	0.3594	0.3632	0.3669	0.3707	0.3745	0.3783
-0.3	0.3821	0.3859	0.3897	0.3936	0.3974	0.4013	0.4052	0.409	0.4129	0.4168
-0.2	0.4207	0.4247	0.4286	0.4325	0.4364	0.4404	0.4443	0.4483	0.4522	0.4562
-0.1	0.4602	0.4641	0.4681	0.4721	0.4761	0.4801	0.484	0.488	0.492	0.496
0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

Table 2: Critical values $t_{\alpha,\nu}$ of t-distribution

	ν										
α	10	11	12	13	14	15	16	17	18	19	20
0.05	1.812	1.796	1.782	1.771	1.761	1.753	1.746	1.74	1.734	1.729	1.725
0.025	2.228	2.201	2.179	2.16	2.145	2.131	2.12	2.11	2.101	2.093	2.086
0.01	2.764	2.718	2.681	2.65	2.624	2.602	2.583	2.567	2.552	2.539	2.528
0.005	3.169	3.106	3.055	3.012	2.977	2.947	2.921	2.898	2.878	2.861	2.845

Table 3: Critical values $\chi^2_{\alpha,\nu}$ of χ^2-distribution

	ν										
α	10	11	12	13	14	15	16	17	18	19	20
0.995	2.156	2.603	3.074	3.565	4.075	4.601	5.142	5.697	6.265	6.844	7.434
0.99	2.558	3.053	3.571	4.107	4.66	5.229	5.812	6.408	7.015	7.633	8.26
0.975	3.247	3.816	4.404	5.009	5.629	6.262	6.908	7.564	8.231	8.907	9.591
0.95	3.94	4.575	5.226	5.892	6.571	7.261	7.962	8.672	9.39	10.12	10.85
0.05	18.31	19.68	21.03	22.36	23.68	25	26.3	27.59	28.87	30.14	31.41
0.025	20.48	21.92	23.34	24.74	26.12	27.49	28.85	30.19	31.53	32.85	34.17
0.01	23.21	24.72	26.22	27.69	29.14	30.58	32	33.41	34.81	36.19	37.57
0.005	25.19	26.76	28.3	29.82	31.32	32.8	34.27	35.72	37.16	38.58	40