Propositional Logic

Fall 2013

Learning Outcomes . . .

At the conclusion of this session, we will

- Define the elements of propositional logic: statements and operations, including implication, and its converse, inverse, and negation.
- Use both truth tables and derivations to demonstrate equivalence of logical statements.
- Translate English expressions into logical statements.
- Define common tautologies, contradictions, and equivalences.
- Recognize and employ *modus ponens* and *modus tollens* and other forms of valid argumentation.

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- Why is it important?
- How will we use it in this class?

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- What is logic?
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- Why is it important?
- Provides a systematic, tractable method of reasoning from given truths (called axiomata or axioms) to new truths (called propositions or theorems).
- How will we use it in this class?
- Logic is the skeleton that supports mathematical truth-making.
- Logic is the glue that holds programs together.

Propositional statements

Definition

A *statement* is a declarative utterance in a language that is either true or false. A statement is either *atomic*, meaning that it contain a single element of truth or falsity, or it is compound, meaning that it consists of statements that are composed with various logical operators.

Example

Let $s :=$ it is raining and $t :=$ the sun is shining. Clearly, s and t are atomic statements. These may be combined in a variety of manners, to be described shortly, but s or t, written $(s \vee t)$ is certainly one conceivable *compound* statement: "it is raining" or "the sun is shining."

Making new statements by connecting propositions

Three fundamental operators are adequate to create the spectrum of logical possibilities.

Operator	**Description**
And (conjunction)	Written $s \wedge t$: true when s *and* t are true.
Or (disjunction)	Written $s \vee t$: true when s *or* t are true.
Negation	Written $\neg s$ (or sometimes $\sim s$ or $\bar{s}$): true only when s is false, and vice versa.

- Keep in mind that s and t may be atomic or compound propositions themselves!

Truth, falsity, and interpretations

- The truth or falsity of any statement depends upon its *context*.
- "Context" can be visualized as the values that are associated with each variable in a statement.

Example

Is $a \vee b$ true? Well, if *either* $a = true$ or $b = true$, then the statement $a \vee b$ is true.
Ask now if $a \wedge b$ is true, and you will see that it is true, but under fewer interpretations—or, its context is different.

Logical equivalences

- How might we determine if two logical statements were equivalent?
  - Construct truth tables for each.
  - Show that one can be transformed into the other through the systematic application of operations—this is sometimes called a *derivation*.
- Your textbook emphasizes constructing truth tables, but mastery of derivations will prove helpful in reading and writing proofs later in this course.

Patterns in truth tables

Truth tables contain rows and columns.

- The number of *necessary rows* is determined by the number of *variables* (this is a homework question!)
- Read rows (horizontal values) as *products*. (Assuming that columns are arranged as they appear in your text: from variables to final forms.)
- Read columns (vertical values) as *co-products* (sums).
- This means that we (generally) care about rows whose terminal values (products) are True.

Apples or Oranges . . . or both!

- Observe that the "or operator" is *inclusive*, meaning $a \vee b$ is true exactly when either a, b or both are true.
- In common English, sometimes, we mean *either a or b*, but *not both*.

Example

Let's construct a new operator, called the "exclusive or," and use both truth tables and derivations (equivalence) to explore its properties.

An important property . . .

Visualize an "exclusive-or" machine that takes two inputs and outputs either a 0 for false and a 1 for true (note: these values are arbitrary).

- If the machine outputs a 1, what do we know about its last input?
- If the machine outputs a 0 . . . ?

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- Would you say that the machine "remembers?"
- Would you say that the machine "computes?" Hint: consider the behavior of our machine on a string of 0 and 1's. If it returns a 0 what might we know about that string, etc.

An effect of negation . . . De Morgan's Laws

Alphonse De Morgan identified a fundamental property that we will find extremely helpful: Let a and b be logical statements, then

$$\neg(a \vee b) = \neg a \wedge \neg b$$

and

$$\neg(a \wedge b) = \neg a \vee \neg b$$

Construct a Truth Table showing De Morgan's Laws

What property does this Law exhibit?

Introducing logical implication

- Arguably, *logical implication* is the most useful operator in predicate logic.
- Logical implication is expressible using only two of the operators introduced in the last slide.
- Implication captures the notion of an action depending upon the success (or failure) of another action.
- Unlike primitive connectors, implication is directional!

Example

If it rains, then Richard brings an umbrella.

We would like a way of assigning truth or falsity to these kinds of statements.

Implication, cont'd.

Examine the truth table for the implication $a \implies b$:

a	**b**	**a $\implies$ b**
T	T	T
F	T	T
T	F	F
F	F	T

Note two important qualities of implication:

1. We see only one case where the implication is false.
2. Compare rows 2 and 3: implication is sensitive to direction!.

Implications of implication . . .

- Everything to the left of the implication symbol is called either the antecedent, the hypothesis, or the **sufficient** condition.
- Everything to the right of the implication symbol is called the consequent, the conclusion, or the **necessary** condition.
- An implication is false just in the case that its hypothesis is true, but its conclusion is false. Said another way: an implication is false when its necessary condition is not satisfied.

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- An implication is false just in the case that its hypothesis is true, but its conclusion is false. Said another way: an implication is false when its necessary condition is not satisfied.
- Unlike in common speech, no relationship need exist between the hypothesis and the conclusion. Thus, implications such as "If the moon is made of cheese, then the empty set is the subset of all sets." are true.

Trying on an implication . . .

Consider the following implication: if $x > 2$, then $x^2 > 4$. Using our understanding of implication, let's see how different interpretations (bindings) for x play out:

1. What happens when $x > 2$? In other words: how does the implication behave when its *hypothesis* is satisfied?

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2. What happens when $x \leq 2$? How does the implication behave when its *hypothesis* is not satisfied?
3. Under what circumstances would the implication ever be false?

Implications, expressed in terms of primitives

- The implication is a *composite* expression built from disjunction and negation:

$$p \implies q \quad \text{has the same truth table as} \quad \neg p \vee q$$

  Try it . . . right now!

- It is crucial that you remember this equivalence because it will prove helpful throughout this (and subsequent) course(s).

Some important variations of implications

We will define the following variations of the implication:

- The "negation" of an implication: $\neg(a \implies b)$.
- The "converse" of an implication?
- The "inverse" of an implication?
- The "contrapositive" of an implication, and show that it is equivalent.
- Discuss equivalences of other forms.

Negating an implication

Simply negating each component of an implication does not negate the implication. Why?

Because the negation of $p \implies q$ is p *and* **not** q! In other words, we need to show that the conclusion does not follow from the premise: in symbols:

$$\neg(p \implies q) \equiv p \wedge \neg q.$$

Show this now by derivation and/or truth-table.

Negate $x > 2 \implies x^2 > 4$.

The converse of an implication

Definition

The *converse* of an implication is obtained by transposing its conclusion with its premise.

Example

Given $p \implies q$, its converse is $q \implies p$.

Construct and evaluate the converse of $x > 2 \implies x^2 > 4$.

The inverse of an implication

Definition

The *inverse* of an implication is obtained by negating both its premise and its conclusion.

Example

Given $p \implies q$, its inverse is $(\neg p) \implies (\neg q)$. (Parentheses added for emphasis.)

Construct and evaluate the inverse of $x > 2 \implies x^2 > 4$.

The Contrapositive form of an implication

Definition

The *contrapositive* form of an implication is an equivalent statement formed by *inverting* its *converse*.

Given $a \implies b$, form its contrapositive as $\neg a \implies \neg b$.

Explore the contrapostitive of $x > 2 \implies x^2 > 4$.

Working through an example on your own...

Construct truth tables and convince yourself that

> *If an object is a square, then it is a closed polygon comprising exactly 4 sides.*

is equivalent to its contrapositive. Show that the original implication is not equivalent to either its converse or its inverse, or their negations.

Is there a relationship between the inverse and the converse of an implication?

Bidirectional implication

Often, we want to state the something happens or that something is true *only when* or *only if* something else is also true.

Definition

The *biconditional* statement is an implication that is true only when its antecedent and its consequent have the *same* truth values; it is false otherwise. In symbols, $p \leftrightarrow q$ is true *only* when $p \implies q$ **and** $q \implies p$. The biconditional is written p **iff** q.

In colloquial speech, “if” is often used when “iff” is logically intended.

See if you can construct its truth table.

Validities & contradictions

- Certain statements are true or false under *any interpretation*. For example: $a \vee \neg a$, is true no matter what truth value is assigned to a. Likewise, $a \wedge \neg a$ is never true, regardless of the truth value assigned to a.
- A statement that is true under any interpretation is called a *tautology* or a *validity*.
- A statement that is false under any interpretation is called a *contradiction*.

Said another way: certain statements are true (or false) by virtue of their logical structure alone—such statements are *formally true* (or *formally false*).

Validity, as a matter of “form.”

Definition

An *argument* is a sequence of statements terminating with a conclusion.

- Validity is based upon *formal* properties, not *content*.
- Mastery of logical argumentation translates into a deeper understanding of and facility for constructing mathematical proof.

Definition (Modus Ponens)

Modus ponens ("To affirm by affirming") is a valid form of *inference* that appears in symbols as:

$$\frac{p \implies q, p}{\therefore q}$$

Informally: if p implies q and we know that p is true, then we may conclude q.

- Commonsense observation: seen in "forward-chaining" production systems.
- Observe that computationally we can view modus ponens as a law of substitution or replacement.

Modus Tollens . . .

Applying the contrapositive, we obtain another form of argumentation:

Definition (Modus Tollens)

Modus tollens ("To affirm by denying") is a valid form of *inference* that appears in symbols as:

$$\frac{p \implies q, \neg q}{\therefore \neg p}$$

Informally: if p implies q and we show q is not the case, then we may conclude that p is not true.

Example

Construct an argument using Modus Tollens using $x > 2 \implies x^2 > 4$.

Recapitulating . . .

- Logic is a system composed of *discrete statements* which are either true or false, but not both.
- Logical Implication ($p \implies q$) is *directional*.
- Bidirectional implication is bidirectional: for example, $p \iff q$ means $(p \implies q) \wedge (q \implies p)$.
- Consistent forms are exactly those logical constructions that are *true* for at least one set of *bindings*.
- A **contradiction** is a logical statement that is false under *any interpretation*.
- A **tautology** is a logical statement that is true under *any interpretation*.

Additional forms commonly used in proofs . . .

- Suppose that we know p is true, then we can argue $p \vee q$. (Generalization)

Example

(Time permitting) Show each of these using our class example.

Additional forms commonly used in proofs . . .

- Suppose that we know p is true, then we can argue $p \vee q$. (Generalization)
- Suppose that we know $p \wedge q$ holds, then was can affirm p, we can also conclude q. (Specialization)

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- Suppose that we know $p \vee q$ holds, *and* we show that $\neg p$, then we can conclude q, and . . . (Elimination)

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- Suppose that we know $p \wedge q$ holds, then was can affirm p, we can also conclude q. (Specialization)
- Suppose that we know $p \vee q$ holds, *and* we show that $\neg p$, then we can conclude q, and . . . (Elimination)
- Suppose that we know $p \implies q$ and $q \implies r$, then we may conclude $p \implies r$. (Transitivity)

Example

(Time permitting) Show each of these using our class example.

How logic fits in

Your success in this class depends upon your ability to construct proofs:

- A "proof" is a collection of definitions, axioms, and conclusions (results of other proofs and lemmas) that convinces the audience of the truth of a proposition (theorem, lemma).
- Logic is the glue that hold these statements together.

Next, we will explore how simple logic underlies programming behaviors, such as flow of control, and arithmetic.