Homework 1: Coordinate Transforms, Rotation Matrices, Homogeneous Transformations

Problems: Due Fri, 9/20
Code: Due week of 9/16, IN LAB

September 16, 2013

Problems:

1. Show that rotations are rigid body transformations, i.e. show that any rotation matrix R preserves the norm of any vector it acts on and the angle between any two vectors it acts on.

2. Consider two bases for $\mathbf{R}^3$

$$\left\{ x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad \left\{ x' = \begin{bmatrix} \sqrt{3}/3 \\ \sqrt{3}/3 \\ \sqrt{3}/3 \end{bmatrix}, y' = \begin{bmatrix} 0 \\ -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}, z' = \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ -1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix} \right\} \tag{1}$$

   - Show that they are both orthonormal.
   - Suppose $v := [v_x \; v_y \; v_z]^T$ is the coordinates of some vector with respect to the standard basis. Find a 3×3 coordinate transformation matrix T such that $v' = Tv$ where v' is the coordinates of the vector written with respect to the $x'y'z'$ basis.
   - Show that T is a rotation matrix and explain briefly why you would expect this.

3. Derive the Rodrigues formula for a rotation matrix R of an angle θ about an axis ω.

$$R = e^{\omega\theta} = I + \hat{\omega}\sin\theta + \hat{\omega}^2(1 - \cos\theta) \tag{2}$$

   (Assume $||\omega|| = 1$.) You may follow either the derivation we did in Discussion (Wed, 9/4) or the derivation in the book. Give some brief intuition for each step.

4. Write down the rotation matrix for an angle of 37.5^o about the axis $[-3, \; 1, \; 2.5]^T$.

5. Find the axis of rotation and angle of rotation of a rotation matrix

$$R = \begin{bmatrix} 3/4 & -1/4 & \sqrt{6}/4 \\ -1/4 & 3/4 & \sqrt{6}/4 \\ -\sqrt{6}/4 & -\sqrt{6}/4 & 1/2 \end{bmatrix}. \tag{3}$$

Code:

Write python functions to implement the following formulas from the book. You will need these to complete the labs starting the week of Mon, 9/16. Some of these formulas, we will cover next week.

- The ($\hat{}$) operator for rotation axes in 3D.
  - Input: 3×1 vector, $\omega = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$.

– Output: 3×3 matrix,

$$\hat{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}. \tag{4}$$

- Rotation matrix in 2D as function of θ.
  – Input: Scalar, θ.
  – Output: 2×2 matrix,

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \tag{5}$$

- Rotation matrix in 3D as a function of ω and θ.
  – Inputs: 3×1 vector, $\omega = \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}^T$, and scalar, θ.
  – Output: 3×3 matrix,

$$R(\omega, \theta) = e^{\hat{\omega}\theta} = I + \frac{\hat{\omega}}{||\omega||} \sin(||\omega||\,\theta) + \frac{\hat{\omega}^2}{||\omega||^2}(1 - \cos(||\omega||\,\theta)) \tag{6}$$

  (This is the form for the Rodrigues formula when $||\omega|| \neq 1$.)

- The ($\hat{}$) operator for twists in 2D.
  – Input: 3×1 vector, $\xi = \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T$. (Note that here ω is a scalar.)
  – Output: 3×3 matrix,

$$\hat{\xi} = \begin{bmatrix} 0 & -\omega & v_x \\ \omega & 0 & v_y \\ 0 & 0 & 0 \end{bmatrix}. \tag{7}$$

- The ($\hat{}$) operator for twists in 3D.
  – Input: 6×1 vector, $\xi = \begin{bmatrix} v^T & \omega^T \end{bmatrix}^T = \begin{bmatrix} v_x & v_y & v_z & \omega_x & \omega_y & \omega_z \end{bmatrix}^T$.
  – Output: 4×4 matrix,

$$\hat{\xi} = \begin{bmatrix} \hat{\omega} & v \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_z & \omega_y & v_x \\ \omega_z & 0 & -\omega_x & v_y \\ -\omega_y & \omega_x & 0 & v_z \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

- Homogeneous transformation in 2D.
  – Inputs: 3×1 vector, $\xi = \begin{bmatrix} v_x & v_y & \omega \end{bmatrix}^T$, and scalar, θ. (Note that here ω is a scalar.)
  – Output: 3×3 matrix,

$$g(\xi, \theta) = e^{\hat{\xi}\theta} = \begin{bmatrix} R & p \\ \mathbf{0} & 1 \end{bmatrix} \tag{9}$$

  where

$$R = \begin{bmatrix} \cos\omega\theta & -\sin\omega\theta \\ \sin\omega\theta & \cos\omega\theta \end{bmatrix} \tag{10}$$

$$p = \begin{bmatrix} 1 - \cos\omega\theta & \sin\omega\theta \\ -\sin\omega\theta & 1 - \cos\omega\theta \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v_x/\omega \\ v_y/\omega \end{bmatrix} \tag{11}$$

- Homogeneous tranformation in 3D.
  - Inputs: 6×1 vector, $\xi = \begin{bmatrix} v^T & \omega^T \end{bmatrix}^T = \begin{bmatrix} v_x & v_y & v_z & \omega_x & \omega_y & \omega_z \end{bmatrix}^T$, and scalar, θ.
  - Output:

$$g(\xi, \theta) = e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \frac{1}{||\omega||^2}(I - e^{\hat{\omega}\theta})(\hat{\omega}v) + \frac{1}{||\omega||^2}\omega\omega^T v\theta \\ 0 & 1 \end{bmatrix} \tag{12}$$

- Product of exponentials in 3D.
  - Inputs: n, 6×1 vectors, $\xi_1, \xi_2, \ldots, \xi_n$, and n, scalars, $\theta_1, \theta_2, \ldots, \theta_n$.
  - Outputs

$$g(\xi_1, \theta_1, \xi_2, \theta_2, \ldots, \xi_n \theta_n) = e^{\hat{\xi}_1\theta_1} e^{\hat{\xi}_2\theta_2} \ldots e^{\hat{\xi}_n\theta_n} \tag{13}$$