Lecture: Homogeneous Transformations, Twists, Screws, Forward Kinematics

Dan Calderone

September 24, 2013

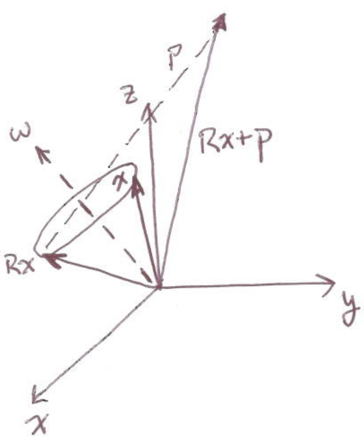
Homogeneous Transformations

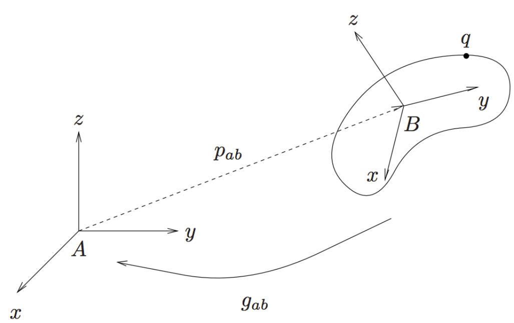
$$g\bar{x} = \begin{bmatrix} R & p \\ \mathbf{0} & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \qquad g \in SE(3) \tag{1}$$

Represents a rotation and then a translation acting on the vector x or a coordinate transformation between two reference frames that differ by a rotation and translation.

Two Uses for Homogeneous Transformations

1. Rotation and translation of a point through space
2. Changing reference frames

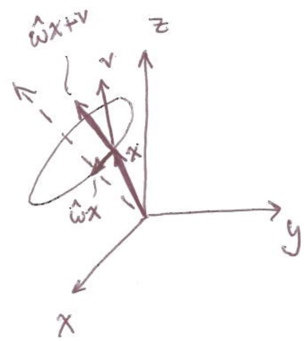




Twists (ξ): Instantaneous translation and rotation

A twist defines the instantaneous motion that produces a homogeneous transformation. i.e. ξ defines an ODE

$$\frac{d}{d\theta}x = \hat{\omega}x + v \qquad \text{i.e.} \qquad \frac{d}{d\theta}\begin{bmatrix} x \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{\omega} & v \\ \mathbf{0} & 0 \end{bmatrix}}_{\hat{\xi}} \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{2}$$



which has solution

$$x(\theta) = e^{\hat{\omega}\theta}x(0) + \int_0^\theta e^{\hat{\omega}(\theta-\tau)}v \; d\tau \tag{3}$$

Given that

$$\frac{d}{d\tau}\left(\hat{\omega}e^{\hat{\omega}(\theta-\tau)} + \omega\omega^T\tau\right) = e^{\hat{\omega}(\theta-\tau)} \tag{4}$$

and v is constant with respect to τ, we have that

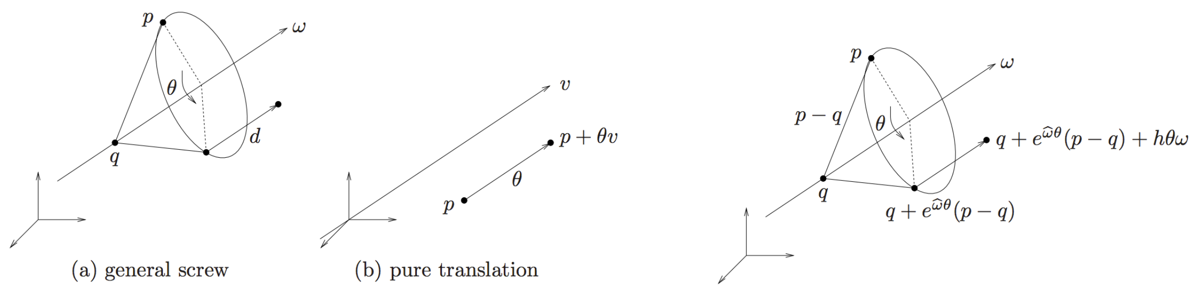
$$x(\theta) = e^{\hat{\omega}\theta}x(0) + \left(I - e^{\hat{\omega}\theta}\right)\hat{\omega}v + \omega\omega^T v\theta \tag{5}$$

Writing this in homogeneous form, we get

$$\begin{bmatrix} x(\theta) \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)\hat{\omega}v + \omega\omega^T v\theta \\ \mathbf{0} & 1 \end{bmatrix}}_{e^{\hat{\xi}\theta}} \begin{bmatrix} x(0) \\ 1 \end{bmatrix} \tag{6}$$

Screw Motion: Rotation about a fixed axis and translation along that same axis

A screw motion is a rotation about a fixed axis (not necessarily through the origin) and a translation along *that same axis.*



(a) general screw (b) pure translation

We can associate every screw motion with a specific twist $\xi = [\ \omega^T \ \ v^T]$.

Relationship between components of a screw and its corresponding twist:

- Pitch:

$$h = \frac{\omega^T v}{|\omega|^2}. \tag{7}$$

  The pitch of a twist is the ratio of translational motion to rotational motion. If $\omega = 0$, we say that ξ has infinite pitch.

- Axis:

$$l = \begin{cases} \{\frac{\omega \times v}{|\omega|^2} + \lambda\omega : \lambda \in \mathbf{R}\}, & \text{if} \quad \omega \neq 0 \\ \{0 + \lambda v : \lambda \in \mathbf{R}\}, & \text{if} \quad \omega = 0 \end{cases} \tag{8}$$

  The axis l is a directed line through a point. For $\omega \neq 0$, the axis is a line in the ω direction going through the point $\frac{\omega \times v}{|\omega|^2}$. For $\omega = 0$, the axis is a line in the v direction going through the origin.

- Magnitude:

$$M = \begin{cases} |\omega|\,, & \text{if} \quad \omega \neq 0 \\ |v|\,, & \text{if} \quad \omega = 0 \end{cases} \tag{9}$$

  The magnitude of a screw is the net rotation if the motion contains a rotational component or the net translation otherwise. If we choose $|\omega| = 1$ (or $|v| = 1$ when $\omega = 0$), then a twist $\hat{\xi}\theta$ has magnitude $M = \theta$.

Twist that generates a screw motion:

$$\hat{\xi} = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix} \quad \text{if} \quad h = \infty \qquad\qquad \hat{\xi} = \begin{bmatrix} \hat{\omega} & -\omega \times q + h\omega \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{if} \quad h \neq \infty \tag{10}$$

$$e^{\hat{\xi}\theta} = \begin{bmatrix} I & \theta v \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{if} \quad h = \infty \qquad\qquad e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right) q + h\theta\omega \\ \mathbf{0} & 1 \end{bmatrix} \quad \text{if} \quad h \neq \infty \tag{11}$$

Robot Manipulators: Forward Kinematics

Twists that generate the screw motions for revolute and prismatic joints:

$$\text{Revolute Joint:} \quad \xi = \begin{bmatrix} -\omega \times q \\ \omega \end{bmatrix} \qquad\qquad \text{Prismatic Joint:} \quad \xi = \begin{bmatrix} v \\ \mathbf{0} \end{bmatrix} \tag{12}$$

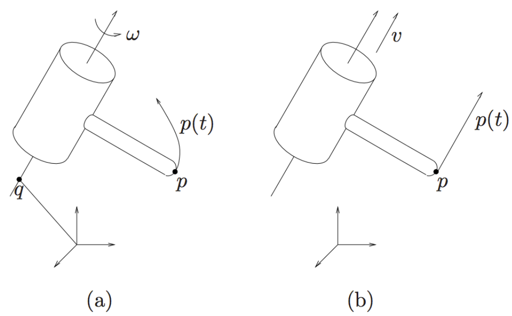


Figure 1: (a) Revolute Joint. (b) Prismatic Joint.

Question: How do I change coordinates from a point in the end effector frame (T) to a point in the base frame (S)?

Product of Exponentials Formula:

$$g_S^T(\theta) = e^{\hat{\xi_1}\theta_1} e^{\hat{\xi_2}\theta_2} \cdots e^{\hat{\xi_n}\theta_n} g_S^T(\mathbf{0}) \tag{13}$$

where $\theta = [\theta_1 \ \ldots \ \theta_n]$ and $\xi_1, \ldots, \xi_n$ are the twists that generate the screw motions of joints 1 through n respectively.