Homework 2: Homogeneous Transformations, Twists, Screws, Forward Kinematics

Due Fri, 10/4, 5 pm

September 27, 2013

Problems:

1. Derive the following equation for a homogeneous transformation resulting from a constant twist, $\xi := [v^T \ \omega^T]$.

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)\hat{\omega}v + \omega\omega^T v\theta \\ \mathbf{0} & 1 \end{bmatrix} \tag{1}$$

   You may follow either the derivation we used in class or the derivation in the book. (If you choose the derivation we did in class, you may use the fact that the solution to the differential equation

$$\dot{x} = \hat{\omega}x + v \tag{2}$$

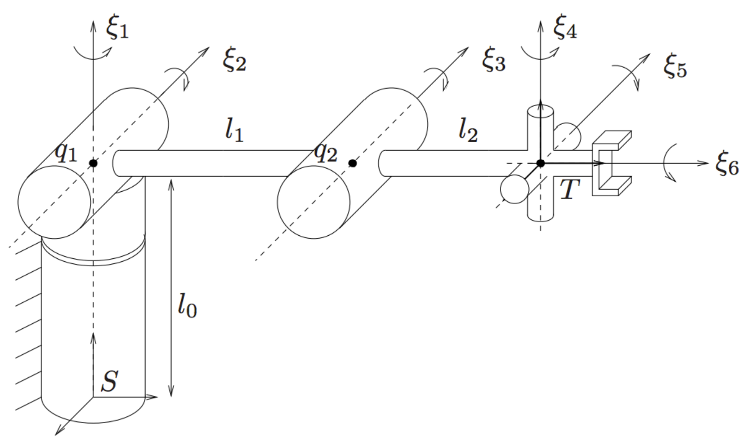
   is given by

$$x(t) = e^{\hat{\omega}t}x(0) + \int_0^t e^{\hat{\omega}(t-\tau)}v \, d\tau \tag{3}$$

   Give a brief physical intuition for what a twist is and how it relates to a homogeneous transformation.

2. True or False: In the forward kinematics formula, the θ's are measured relative to the previous link. Briefly explain your answer.

3. Write down the forward kinematics for the following manipulator.



   Write out the twist that generates the proper screw motion for each of the joints given the following link lengths.

l_0	l_1	l_2
1.5 m	0.75 m	0.5 m

Then, compute the transformation from the tool frame to the base frame for each of the following sets of θ's. (You can use the code you wrote last week.)

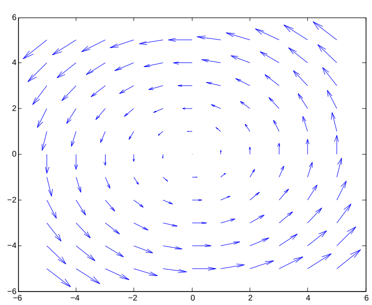
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
Case 1	$\pi/4$	$-\pi/4$	$\pi/4$	$-\pi/4$	0	$\pi/2$
Case 2	$-\pi/4$	$-\pi/4$	$-\pi/4$	0	$\pi/2$	$\pi/4$

4. Rotations, Twists, and Screws
   The purpose of this problem is to help you visualize how a rotation matrix is generated by an axis of rotation ω and an angle θ, how a homogeneous transformation is generated by a twist ξ and a parameter θ, and to illustrate the relationship between twists and screws. You will need to simulate several things. You may use either Python or Matlab. If you use Matlab, you will find the functions `quiver` and `quiver3` useful. If you use Python, you will find the function `matplotlib.pyplot.quiver()` useful for 2D plots. There is also some code on Piazza that will help you plot vector fields in 3D.

   - Rotations in 2D:
     - Plot the vector field given by

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \tag{4}$$

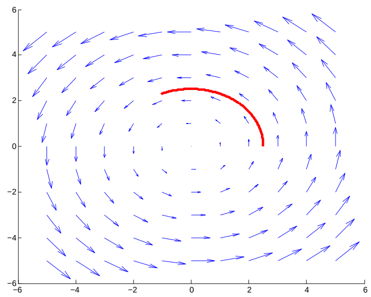
       Your plot should show a set of arrows over a grid of points in the plane. For each point x, the arrow should be defined by $\dot{x}$ given above. Your plot should look like the following.



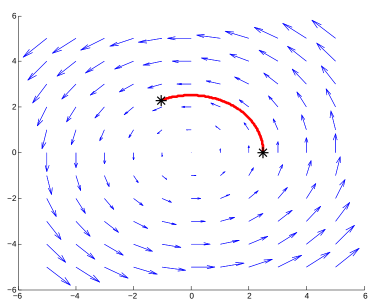
     - Integrate the vector field starting from initial condition $x(0) = [2.5\ 0]^T$ at time 0 to some final position $x(\theta)$ at time θ (you may choose θ); that is calculate a sequence of points that show how x evolves over time starting from $x(0)$ at time 0 to $x(\theta)$ at time θ. x at the next time step should be given by

$$x(t + \Delta t) = x(t) + \Delta t \cdot \dot{x}(t) \tag{5}$$

       Plot this trajectory on top of the vector field. Your plot should look like the following.



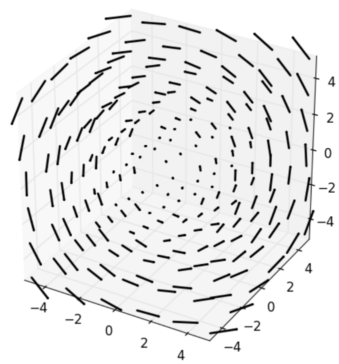
– Calculate the rotation matrix, $e^{\hat{\omega}\theta}$. Plot $x(0)$ and $x(\theta) = e^{\hat{\omega}\theta}x(0)$ on the same plot. Your plot should look something like this.



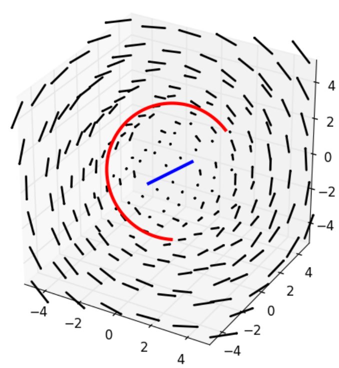
- Rotations in 3D:
  – Plot the vector field given by

$$\dot{x} = \hat{\omega}x \tag{6}$$

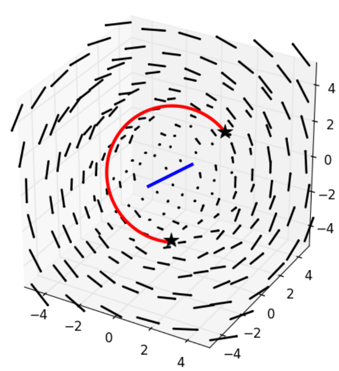
for $\omega = [\ 0.5774 \ \ -0.5774 \ \ 0.5774]^T$. Your plot should look like the following.



– Integrate the vector field starting from the initial condition $x(0) = [2\ 2\ 2]^T$. Also, plot the axis of rotation.



– Calculate the rotation matrix, $e^{\hat{\omega}\theta}$. Plot $x(0)$ and $x(\theta) = e^{\hat{\omega}\theta}x(0)$ on the same plot.



- Homogeneous Transformations in 3D:
  – Plot the vector field given by

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \hat{\xi} \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{7}$$

for $\xi = [1 \;\; 1 \;\; 1 \;\; 0.5774 \;\; -0.5774 \; 0.5774]^T$.

- Compute the axis of the screw motion generated by the above twist and plot it. Also, plot the origin and compute the pitch of the screw motion.
- Integrate the vector field starting from several different initial conditions. Make sure you choose at least one on the axis of the screw. Comment on how the trajectories differ depending on how close the initial conditions are to the axis of the screw.
- Compute the homogeneous transformation, $e^{\hat{\xi}\theta}$. Plot $x(0)$ and $x(\theta) = e^{\hat{\xi}\theta}x(0)$ for each of the trajectories that you computed before. $x(0)$ and $x(\theta)$ should line up with the beginning and end of each of the trajectories.