# Homework 2: Homogeneous Transformations, Twists, Screws, Forward Kinematics

Due Fri, 10/4, 5 pm

October 8, 2013

## Problems:

1. Derive the following equation for a homogeneous transformation resulting from a constant twist, $\xi := [v^T \ \omega^T]$.

$$e^{\hat{\xi}\theta} = \begin{bmatrix} e^{\hat{\omega}\theta} & \left(I - e^{\hat{\omega}\theta}\right)\hat{\omega}v + \omega\omega^T v\theta \\ \mathbf{0} & 1 \end{bmatrix} \tag{1}$$

   You may follow either the derivation we used in class or the derivation in the book. (If you choose the derivation we did in class, you may use the fact that the solution to the differential equation

$$\dot{x} = \hat{\omega}x + v \tag{2}$$

   is given by

$$x(t) = e^{\hat{\omega}t}x(0) + \int_0^t e^{\hat{\omega}(t-\tau)}v \ d\tau \tag{3}$$

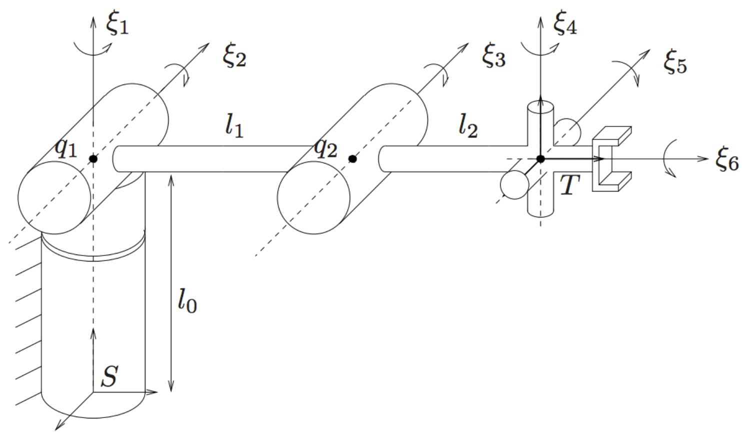
   Give a brief physical intuition for what a twist is and how it relates to a homogeneous transformation.
   **Solution:**
   The two derivations can be found in discussion notes from Sept, 11 and in the book Ch. 2. Sec. 3.2 p. 42.
   The intuition for a twist is that it defines a vector field

$$\dot{\bar{x}} = \hat{\xi}\bar{x} \tag{4}$$

   Flowing along this vector field produces the motion described by the corresponding homogeneous transformation.

2. True or False: In the forward kinematics formula, the $\theta$'s are measured relative to the previous link. Briefly explain your answer.
   **Solution:**
   This is true. From the derivation of the forward kinematics formula we see that each of the $\theta$'s parametrizes a screw motion around one of the axes in the 0-configuration. We can think of the forward kinematics formula as starting with a point $q_T$ in the tool frame, transforming it into the base frame as if the arm were in the 0-configuration, $g_{ST}(0)q_T$, and then applying a screw motion from each of the joints starting with the joints closest to the hand and then moving backwards up toward the shoulder. First, $e^{\hat{\xi}_n\theta_n}g_{ST}(0)q_T$, then $e^{\hat{\xi}_{n-1}\theta_{n-1}}e^{\hat{\xi}_n\theta_n}g_{ST}(0)q_T$, etc. Thus $\theta_i$ is measured from the previous link in the 0-configuration. When the other joint angles higher up the chain are applied, they don't change the value of $\theta_i$ which is still measured with respect to the previous link. For a clearer discussion, see the discussion notes on forward kinematics from Sept 17 (on Piazza).

3. Write down the forward kinematics for the following manipulator.




Write out the twist that generates the proper screw motion for each of the joints given the following link lengths.

| $l_0$ | $l_1$ | $l_2$ |
|---|---|---|
| 1.5 m | 0.75 m | 0.5 m |

Then, compute the transformation from the tool frame to the base frame for each of the following sets of $\theta$'s. (You can use the code you wrote last week.)

| | $\theta_1$ | $\theta_2$ | $\theta_3$ | $\theta_4$ | $\theta_5$ | $\theta_6$ |
|---|---|---|---|---|---|---|
| Case 1 | $\pi/4$ | $-\pi/4$ | $\pi/4$ | $-\pi/4$ | 0 | $\pi/2$ |
| Case 2 | $-\pi/4$ | $-\pi/4$ | $-\pi/4$ | 0 | $\pi/2$ | $\pi/4$ |

**Solution:**
Since all the joints of the above manipulator are revolute joints, we can calculate the twists according to the formula

$$\xi_i = \begin{bmatrix} -\omega_i \times q_i \\ \omega_i \end{bmatrix} \tag{5}$$

where $\omega_i$ is the direction of the axis of rotation of joint $i$ (scaled so that the $||\omega_i|| = 1$) and $q_i$ is any point on the axis. Remember $\omega_i$ and $q_i$ are specified in the 0-configuration.

$$\omega_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \omega_2 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \omega_5 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \quad \omega_6 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{6}$$

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad q_2 = \begin{bmatrix} 0 \\ 0 \\ 1.5 \end{bmatrix} \quad q_3 = \begin{bmatrix} 0 \\ 0.75 \\ 1.5 \end{bmatrix} \quad q_4 = \begin{bmatrix} 0 \\ 1.25 \\ 1.5 \end{bmatrix} \quad q_4 = \begin{bmatrix} 0 \\ 1.25 \\ 1.5 \end{bmatrix} \quad q_4 = \begin{bmatrix} 0 \\ 1.25 \\ 1.5 \end{bmatrix} \tag{7}$$

Note that $q_1$ and $q_2$ are NOT the $q_1$ and $q_2$ in the diagram.

$$\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_2 = \begin{bmatrix} 0 \\ -1.5 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_3 = \begin{bmatrix} 0 \\ -1.5 \\ 0.75 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_4 = \begin{bmatrix} 1.25 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \xi_5 = \begin{bmatrix} 0 \\ -1.5 \\ 1.25 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad \xi_6 = \begin{bmatrix} -1.5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{8}$$

For the two cases, from the product of exponentials formula, $g_{ST}(\theta) = e^{\hat{\xi}_1\theta_1} \cdots e^{\hat{\xi}_6\theta_6} g_{ST}(0)$, we have that

$$\text{Case 1:} \quad g_{ST}(\theta) = \begin{bmatrix} 0 & 0 & 1 & -0.7286 \\ 0 & 1 & 0 & 0.7286 \\ -1 & 0 & 0 & 2.0303 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{Case 2:} \quad g_{ST}(\theta) = \begin{bmatrix} 0.5 & 0.7071 & 0.5 & 0.3750 \\ -0.5 & 0.7071 & -0.5 & 0.3750 \\ -0.7071 & 0 & 0.7071 & 2.5303 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{9}$$

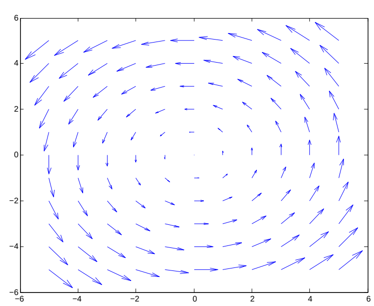
4. Rotations, Twists, and Screws

The purpose of this problem is to help you visualize how a rotation matrix is generated by an axis of rotation $\omega$ and an angle $\theta$, how a homogeneous transformation is generated by a twist $\xi$ and a parameter $\theta$, and to illustrate the relationship between twists and screws. You will need to simulate several things. You may use either Python or Matlab. If you use Matlab, you will find the functions `quiver` and `quiver3` useful. If you use Python, you will find the function `matplotlib.pyplot.quiver()` useful for 2D plots. There is also some code on Piazza that will help you plot vector fields in 3D.

- Rotations in 2D:
  - Plot the vector field given by

$$\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x \tag{10}$$

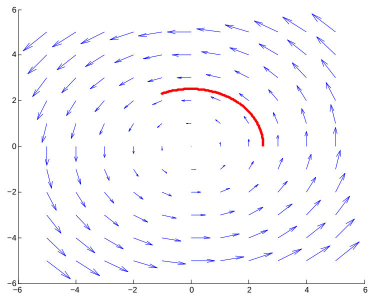
Your plot should show a set of arrows over a grid of points in the plane. For each point $x$, the arrow should be defined by $\dot{x}$ given above. Your plot should look like the following.



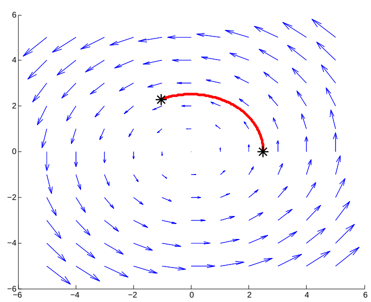

  - Integrate the vector field starting from initial condition $x(0) = [2.5 \; 0]^T$ at time 0 to some final position $x(\theta)$ at time $\theta$ (you may choose $\theta$); that is calculate a sequence of points that show how $x$ evolves over time starting from $x(0)$ at time 0 to $x(\theta)$ at time $\theta$. $x$ at the next time step should be given by

$$x(t + \Delta t) = x(t) + \Delta t \cdot \dot{x}(t) \tag{11}$$

Plot this trajectory on top of the vector field. Your plot should look like the following.



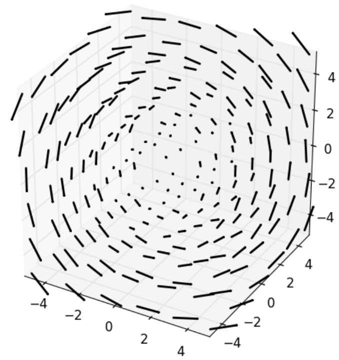

– Calculate the rotation matrix, $e^{\hat{\omega}\theta}$. Plot $x(0)$ and $x(\theta) = e^{\hat{\omega}\theta}x(0)$ on the same plot. Your plot should look something like this.



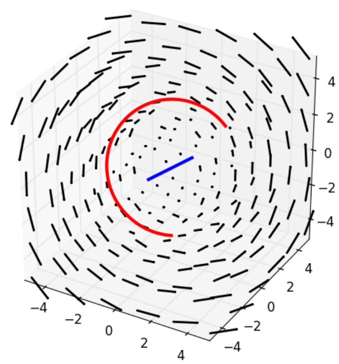

- Rotations in 3D:
  - Plot the vector field given by

$$\dot{x} = \hat{\omega}x \tag{12}$$

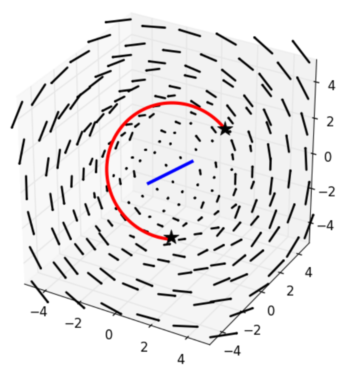
for $\omega = [\ 0.5774\ \ -0.5774\ \ 0.5774]^T$. Your plot should look like the following.



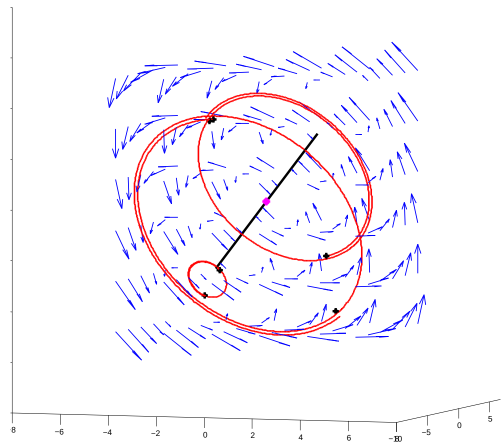
- Integrate the vector field starting from the initial condition $x(0) = [2\ 2\ 2]^T$. Also, plot the axis of rotation.



- Calculate the rotation matrix, $e^{\hat{\omega}\theta}$. Plot $x(0)$ and $x(\theta) = e^{\hat{\omega}\theta}x(0)$ on the same plot.



**Comment:**
It should be noted that in the case of a pure rotation the axis of rotation always passes through the origin. Also for large enough $\theta$ the trajectories form a loop around the axis. These facts can be seen in the plot below. The origin is plotted in purple. (Note that the trajectories don't exactly line up with each other after they travel around the axis once due to numerical error.)



- Homogeneous Transformations in 3D:
  - Plot the vector field given by

$$\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix} = \hat{\xi} \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{13}$$

    for $\xi = [1 \;\; 1 \;\; 1 \;\; 0.5774 \;\; -0.5774 \; 0.5774]^T$.
  - Compute the axis of the screw motion generated by the above twist and plot it. Also, plot the origin and compute the pitch of the screw motion.

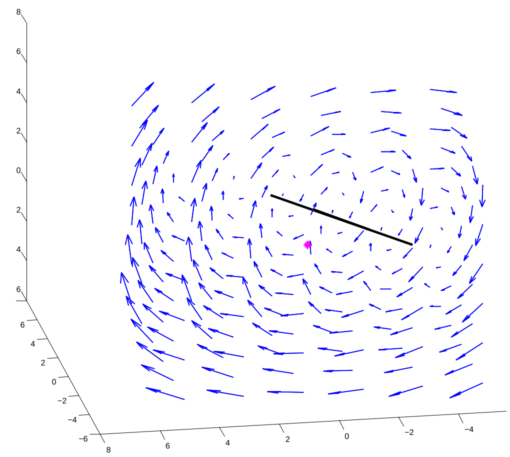
    **Solution:**

    The axis of the screw can be computed as a line in the direction of $\omega$ through the point $q$ where

$$q = \frac{\omega \times v}{||\omega||^2} \tag{14}$$

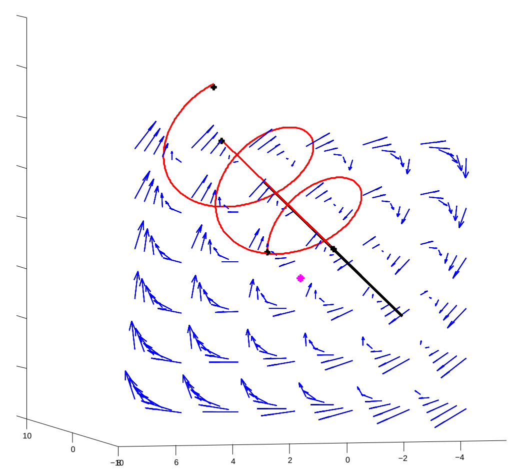
    In the plot, the axis of the screw is shown in black and the origin is shown in purple. Note that in the case of a general homogeneous transformation, the axis of the screw does not have to pass through the origin. The pitch of the screw, $h$ is the magnitude of the projection of $v$ onto the axis of the screw.

$$h = \frac{\omega^T v}{||\omega||^2} \tag{15}$$



– Integrate the vector field starting from several different initial conditions. Make sure you choose at least one on the axis of the screw. Comment on how the trajectories differ depending on how close the initial conditions are to the axis of the screw.

– Compute the homogeneous transformation, $e^{\hat{\xi}\theta}$. Plot $x(0)$ and $x(\theta) = e^{\hat{\xi}\theta}x(0)$ for each of the trajectories that you computed before. $x(0)$ and $x(\theta)$ should line up with the beginning and end of each of the trajectories.
**Solution:**
We integrate from two initial conditions: one on the axis of the screw and one off the axis of the screw.



The trajectory starting on the axis simply moves straight down the axis and is unaffected by the rotational component. The trajectory starting off the axis spirals around the axis but unlike in the case of a pure rotation, the trajectory does not double back on itself but rather travels along the axis in a 'screw' shaped motion. This picture of a screw motion is the general picture we would get if we plotted the vector field from any twist, $\xi = [v^T \ \omega^T]^T$. It is quite interesting and not obvious that every twist should produce this kind of screw motion about an axis somewhere in space. The properties of the screw motion depend on the relationship between $v$ and $\omega$. The component of $v$ that points in the direction of $\omega$ causes the translation along the axis of the screw. The component of $v$ that is orthogonal to $\omega$ serves to shift the axis of rotation away from the origin (recall the formulas for $q$ and $h$ above).