COT 3100, Final Exam (2 hours). Dec 13, 2010

Name (no ID please): ______________________________

There are 6 problems on 6 pages. Be sure you have all the pages.

1. (15 points) Use logical equivalences to show that the following statement is a tautology. Do not use a truth table.

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

Solution: Here is one method.

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

$$\neg[\neg p \wedge (p \vee q)] \vee q$$

$$[p \vee \neg(p \vee q)] \vee q$$

$$(p \vee q) \vee \neg(p \vee q)$$

$$T$$

2. (15 points)
Consider the following function $f : \mathbb{N} \to \mathbb{N}$, and recall that $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

(a) (5 points) Is the function onto? Explain.

Solution: Yes. There are at least 2 ways to explain this.
If $y \geq 0$ is even, then $y + 1 \geq 1$ is odd, so $f(y + 1) = (y + 1) - 1 = y$.
If $y \geq 1$ is odd, then $y - 1 \geq 0$ is even, so $f(y - 1) = (y - 1) + 1 = y$.
In both cases, for any y we have an x so that $f(x) = y$.

You can also explain this by drawing a diagram or graph of the function, and explaining that all y have at least one x for which $f(x) = y$.

(b) (5 points) Is the function one-to-one? Explain.

Solution: Yes. There are at least 2 ways to explain this.

If n is even, then $f(n)$ is odd, and $g(n) = n + 1$ is one-to-one. Likewise, if n is odd, then $f(n)$ is even, and $g(n) = n - 1$ is one-to-one. The case for n odd/even and $f(n)$ even/odd do not overlap, so there cannot be two values x_1 and x_2 such that $f(x_1) = f(x_2)$.

You can also explain this by drawing a diagram or graph of the function, and explaining for any y, there is at most one x for which $f(x) = y$.

(c) (5 points) If the function has an inverse, state what it is. If it does not have an inverse, explain why.

Solution: The function itself is its own inverse.

(d) (x points) Show that $f(n)$ is $O(n)$. Be specific (find the constants c and k).

Solution: We need to show that

$$0 \leq f(n) \leq cn$$

for all $n \geq k$, for some constants c and k. There are many ways to do this.
$f(n) \leq n + 1$ for all n, and $f(n) \geq 0$ for all n. So we have

$$0 \leq f(n) \leq n + 1$$

Pick $k = 1$, and then assume $n \geq k$. So we have

$$0 \leq f(n) \leq n + 1 \leq n + n = 2n = cn$$

for $c = 2$.

3. (15 points) A standard deck of cards has 52 cards, with 13 of each of four suits (clubs, spades, hearts, and diamonds). Clubs and spades are black. Hearts and diamonds are red. Suppose you select cards one at a time from a deck of cards, in some arbitrary (random) order. Assume that you cannot look at the cards you have selected until you have selected all of them.

   How many cards must you select to guarantee that ...

   (a) (3 points) ... at least 3 cards of a same color (red or black) are chosen?

   Solution: This is a generalized pigeon-hole problem. We have $k = 2$ pigeon-holes, and we need to pick N cards so that $\lceil N/2 \rceil = 3$. The solution is $N = 5$.

   (b) (3 points) ... at least 3 red cards are chosen?

   Solution: This is *not* a pigeon-hole problem. To get at least 3 red cards, we need to select $26 + 3 = 29$ cards. If we pick 26, we might get all black cards, so we need to pick 3 more to get at least 3 red cards.

   (c) (3 points) ... at *most* 3 cards of a same color are chosen?

   Solution: Again, this is *not* a pigeon-hole problem. To ensure we get at most 3 cards, we can pick at most 3 cards. If we pick 4 cards, we might get 4 cards of the same color.

   (d) (3 points) ... at least 3 cards from a single suit are chosen?

   Solution: This is a generalized pigeon-hole problem. We have $k = 4$ pigeon-holes, and we need to pick N cards so that $\lceil N/4 \rceil = 3$. The solution is $N = 9$.

   (e) (3 points) ... at least 3 clubs are chosen?

   Solution: This is *not* a pigeon-hole problem. To get at least 3 clubs cards, we need to select $3 \times 13 + 3 = 42$ cards. If we pick 39, we might get all the hearts, spades, and diamonds, and no clubs at all. So we need to pick 3 more to get at least 3 clubs.

4. (20 points) Let p_n be the number of permutations of a set of n elements.

(a) (4 points) What is $P(n, n)$, the number of n-permutations of a set of size n?
Solution: $P(n, n) = n!$.

(b) (8 points) Write a recurrence relation for p_n in terms of p_{n-1}. Be sure to state the base case.
Solution: $p_n = np_{n-1}$, base case is $p_1 = 1$ or $p_0 = 1$ (either base case is fine)..

(c) (8 points) Show, via induction, that the solution to your recurrence is $p_n = P(n, n)$, the number of n-permutations of a set of size n. Don't forget the base case.
Solution: Base case: $p_n = n!$ so $p_1 = 1! = 1$, or $p_0 = 0! = 1$.
Inductive step. Assume $p_{n-1} = (n-1)!$ and show that $p_n = n!$.
If $p_{n-1} = (n-1)!$, then $p_n = np_{n-1} = n((n-1)!) = n!$.

5. (15 points) Consider an equivalence relation R on the integers 1 through 8, with two equivalence classes, one containing $\{1, 2, 3, 4\}$ and another with $\{5, 6, 7, 8\}$. Can $(4, 5)$ be in R? Why or why not? If it is possible, give an example. If it is not possible, explain why. Be precise.

   Solution: It is not possible (a proof by contradiction).

   If R is an equivalence relation, then it is reflexive, symmetric, and transitive.

   Since the integers 1 through 4 are all pairwise related, for any integer a from 1 to 4, we have $(a, 4) \in R$. So if $(4, 5) \in R$, we have $(a, 5) \in R$ by transitivity. We have $(5, b) \in R$ for any integer b from 5 to 8, so by transitivity, we have $(a, b) \in R$.

   Likewise, $(5, 4) \in R$ because of symmetry, so if we swap a and b in the paragraph above, we have $(b, a) \in R$.

   Thus, if $(4, 5) \in R$, then all elements in the set are related to one another, and the two equivalence classes collapse into a single class of 8 integers. This is a contradiction.

6. (20 points) Consider a complete undirected graph G with 5 vertices (1,2,3,4,5) and with all self-edges included.

   (a) (2 points) How many vertex-induced subgraphs does G have?
   Solution: 2^5

   (b) (4 points) Prove or disprove the following statement:

   All vertex-induced subgraphs of G are connected graphs.

   Solution: They are connected. Selecting a single vertex gives a connected graph of one vertex (the self-edge is in the subgraph). Anyway, all graphs of one vertex are connected, whether they have a self-edge or not.

   If more than one vertex is selected, consider any pair a and b selected. Since G is a complete graph, the edge (a, b) will be in the subgraph. So there is a path from a to b (a single edge) in the subgraph. So the subgraph is connected.

   (c) (3 points) How many edges does G have? (Careful, the answer is not 25).
   Solution: G has 15 edges.

   (d) (3 points) How many edge-induced subgraphs does G have?
   Solution: 2^{15}

   (e) (4 points) Prove or disprove the following statement:

   All edge-induced subgraphs of G are connected graphs.

   Solution: This is false. For example, select edges (1,2) and (3,4). There are 4 vertices (1,2,3,4), and two connected components, with no path from 2 to 3.

   (f) (4 points) Draw a subgraph of G that is neither edge-induced nor vertex-induced, or prove that there is no such subgraph.

   Solution: There are many such subgraphs. The simplest one is a graph with vertices 1 and 2, and no edges at all.

   This is not a vertex-induced subgraph. If it were, we would pick vertices 1 and 2 and then we must pick edges (1,1), (1,2), and (2,2).

   This is not an edge-induced subgraph, since it has no edges at all. If we pick no edges, we get no vertices in our subgraph, and the subgraph is empty.