COT 3100, Final Exam (2 hours). Dec 13, 2010

Name (no ID please): ______________________________

There are 6 problems on 6 pages. Be sure you have all the pages.

1. (15 points) Use logical equivalences to show that the following statement is a tautology. Do not use a truth table.

$$[\neg p \wedge (p \vee q)] \rightarrow q$$

2. (15 points)

Consider the following function $f : \mathbb{N} \to \mathbb{N}$, and recall that $\mathbb{N} = \{0, 1, 2, 3, ...\}$.

$$f(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

(a) (5 points) Is the function onto? Explain.

(b) (5 points) Is the function one-to-one? Explain.

(c) (5 points) If the function has an inverse, state what it is. If it does not have an inverse, explain why.

3. (15 points) A standard deck of cards has 52 cards, with 13 of each of four suits (clubs, spades, hearts, and diamonds). Clubs and spades are black. Hearts and diamonds are red. Suppose you select cards one at a time from a deck of cards, in some arbitrary (random) order. Assume that you cannot look at the cards you have selected until you have selected all of them.

   How many cards must you select to guarantee that ...

   (a) (3 points) ... at least 3 cards of a same color (red or black) are chosen?

   (b) (3 points) ... at least 3 red cards are chosen?

   (c) (3 points) ... at *most* 3 cards of a same color are chosen?

   (d) (3 points) ... at least 3 cards from a single suit are chosen?

   (e) (3 points) ... at least 3 clubs are chosen?

4. (20 points) Let p_n be the number of permutations of a set of n elements.

   (a) (4 points) What is $P(n, n)$, the number of n-permutations of a set of size n?

   (b) (8 points) Write a recurrence relation for p_n in terms of p_{n-1}. Be sure to state the base case.

   (c) (8 points) Show, via induction, that the solution to your recurrence is $p_n = P(n, n)$, the number of n-permutations of a set of size n. Don't forget the base case.

5. (15 points) Consider an equivalence relation R on the integers 1 through 8, with two equivalence classes, one containing $\{1, 2, 3, 4\}$ and another with $\{5, 6, 7, 8\}$. Can $(4, 5)$ be in R? Why or why not? If it is possible, give an example. If it is not possible, explain why. Be precise.

6. (20 points) Consider a complete undirected graph G with 5 vertices (1,2,3,4,5) and with all self-edges included.

   (a) (2 points) How many vertex-induced subgraphs does G have?

   (b) (4 points) Prove or disprove the following statement:

   All vertex-induced subgraphs of G are connected graphs.

   (c) (3 points) How many edges does G have? (Careful, the answer is not 25).

   (d) (3 points) How many edge-induced subgraphs does G have?

   (e) (4 points) Prove or disprove the following statement:

   All edge-induced subgraphs of G are connected graphs.

   (f) (4 points) Draw a subgraph of G that is neither edge-induced nor vertex-induced, or prove that there is no such subgraph.