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Physics 356F: Problem Set #4 assigned 14 November 2013 (due the week of 27 November at the start of tutorial)

Note: although you have two weeks to complete this, I recommend you start early, as I plan to assign one final problem set next week.

1. Two energy-eigenstates of the harmonic oscillator, $|m\rangle$ and $|n\rangle$, have a nonvanishing matrix element of X^2, i.e., $\langle m|X^2|n\rangle \neq 0$.

   (a) Write down how to express this fact in the position basis, in terms of $\psi_m(x)$ and $\psi_n(x)$. (You do not need to solve the resulting equation!)

   (b) Expanding X^2 in terms of a and $a^\dagger$ (*instead* of working in the position basis), what can you conclude about m and n?

2. If two potential wells are separated by a repulsive delta-function barrier, the two lowest energy states can be shown to be symmetric and antisymmetric superpositions of the ground states $|L\rangle$ and $|R\rangle$ of the left and right wells, $|S\rangle \equiv \frac{1}{\sqrt{2}}(|L\rangle + |R\rangle)$ and $|A\rangle \equiv \frac{1}{\sqrt{2}}(|L\rangle - |R\rangle)$.
   Suppose that

   $$\begin{aligned} \mathcal{H}|S\rangle &= E_S|S\rangle \text{ and} \\ \mathcal{H}|A\rangle &= E_A|A\rangle \ . \end{aligned} \tag{1}$$

   If at $t = 0$ the particle is in state $|L\rangle$, solve for the state as a function of time.
   Calculate $|\langle L|\Psi(t)\rangle|^2$.

3. Consider two spin-1/2 particles in a Hamiltonian $\mathcal{H} = aS_{x1}S_{x2}$. At $t = 0$, they are in the state $|\Psi(0)\rangle = |\uparrow\rangle|\uparrow\rangle$.

   (a) Write down four orthogonal eigenstates of $\mathcal{H}$ and the corresponding eigenvalues. Hint: since the Hamiltonian commutes with S_{x1} and S_{x2}, which also commute with one another, all three operators may be simultaneously diagonalized.

   (b) Write down $|\Psi(0)\rangle$ in terms of these eigenstates.

   (c) Given your knowledge of the time evolution of the stationary states, you can write down $|\Psi(t)\rangle$ easily now, using the superposition principle. (Do so.)

   (d) At time t, suppose particle 1 is found in state $|+x\rangle$. What is the state of particle 2? Describe what this means (physically, in what direction is particle 2 pointing?).

(e) At time t, suppose particle 1 is found in state $|-x\rangle$. What is the state of particle 2? Describe what this means (physically, in what direction is particle 2 pointing?).

(f) Given your knowledge of how particle 2 would behave in a static magnetic field (section 4.3), explain the results of d and e in one or two sentences.

(g) Use the fact that $\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle[\mathcal{H}, A]\rangle$ for operators without explicit time-dependence (equation 4.16 from the text) to calculate how $\langle S_{z1}\rangle$ and $\langle S_{y1}\rangle$ vary as a function of time for a given value of S_{x2} (for this part alone, do not use the initial state provided for the other parts of the problem).

(h) Calculate the probabilities of finding the spins in $|\uparrow\rangle|\uparrow\rangle$, $|\downarrow\rangle|\downarrow\rangle$, and $|\uparrow\rangle|\downarrow\rangle$ as a function of time.

(i) What quantity is conserved according to the results of part h? (Hint: it is bilinear in spin operators.)

(j) Demonstrate from the commutation laws that you should have expected this conservation law from the start.

4. For each of the following potentials $U(x, y)$, *explain* whether they are separable in Cartesian coordinates, cylindrical coordinates, both, or neither.

   (a) $x + y$

   (b) $e^{-(x^2+y^2)/2r^2}$

   (c) 0 iff $|x| < r$ and $|y| < r$; V_0 otherwise.

   (d) $\sqrt{x^2 + y^2}$

5. Consider a two-dimensional harmonic oscillator (the three-dimensional case is treated in section 10.5). We can write the state $\psi(x, y) = \psi_1(x)\psi_2(y)$ or $|\Psi\rangle = |\psi_1\rangle_x|\psi_2\rangle_y$. The Hamiltonian is $\mathcal{H} = (P_x^2 + P_y^2)/2m + (X^2 + Y^2)m\omega^2/2$, which can be separated into $\mathcal{H} = H_x + H_y$ in the obvious way. We define our x- and y-eigenstates as usual, by $H_i|n\rangle_i = (n + 1/2)\hbar\omega|n\rangle_i$, where $i = \{x, y\}$.
   We define 1D lowering operators $a_y = \sqrt{\frac{m\omega}{2\hbar}}\left(Y + \frac{i}{m\omega}P_y\right)$ and the same for x.
   This allows us to write $H_x = \hbar\omega\left(a_x^\dagger a_x + 1/2\right)$ and the same for y.

   (a) What are the energies of the states $|0\rangle_x|0\rangle_y$, $|0\rangle_x|1\rangle_y$, and $|1\rangle_x|1\rangle_y$?

   (b) Use the commutators to find $d\langle a_x\rangle/dt$ and $d\langle a_y\rangle/dt$ (although the lowering operators are not Hermitian observables, one may still calculate their time dependence in this way).

   (c) At what frequenc(y)(ies) can observables evolve if the system is initially prepared in state $(|01\rangle - |10\rangle)/\sqrt{2}$?

(d) Let us define new operators $a_{\pm 45} = (a_x \pm a_y)/\sqrt{2}$ to separate the potential along the 45 and -45 axes instead of x and y.
Rewrite the Hamiltonian in terms of these operators.

(e) What is the action of these operators on the state $\left(|01\rangle - |10\rangle\right)/\sqrt{2}$? What do you conclude about the physical meaning of this state?