(L2) - Basic properties of pseudoholomorphic curves

almost complex manifolds and Riemann surfaces

M manifold, $J = (J(p) : T_pM \to T_pM)_{p\in M}$ almost complex structure : $J^2 = -id$

Integrability thm: (M, J) is complex (locally $\cong (\mathbb{C}^n, i = J_0)$)

$\Longleftrightarrow \underbrace{[JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]}_{N_J(X,Y)} = 0 \quad \forall X, Y : M \to TM$ vector field

$N_J(X,Y)$ Nijenhuis tensor

Corollary (Homework: $\dim_{\mathbb{R}} M = 2 \Rightarrow N_J = 0$ always $\Rightarrow$)

Every almost complex 2-manifold (Σ, j) is complex (locally $\simeq (\mathbb{C}, i)$).

Uniformization thm: Every complex structure j on S^2 is equivalent

to $j_0 = i$ on $\mathbb{CP}^1$ (i.e. $\exists \varphi : \mathbb{CP}^1 \longrightarrow S^2$ diffeom. : $\varphi^* j = j_0$)

Pseudoholomorphic curves

Def^n: (Σ, j), (M, J) almost complex

- A submanifold $C \subset M$ is J-holomorphic if $J(TC) = TC$.
- A map $u : \Sigma \to M$ is (J, j)-holomorphic if $J \circ du = du \circ j$

$\Leftrightarrow \quad \bar{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) = 0$ "Cauchy-Riemann operator"

Uniformization
Cor: $C \subset M$ J-holomorphic submfd, $C \simeq S^2$

$$\Leftrightarrow \exists\, u : (\mathbb{CP}^1, j_0) \hookrightarrow M \ : \ \bar{\partial}_J u = 0 \ , \ u(\mathbb{CP}^1) = C$$

Proof: $u: \quad \underset{\phi^* \iota^* J}{\overset{\shortparallel}{(\mathbb{CP}^1, j_0)}} \xrightarrow{\varphi} (\underset{\text{2 dim}}{C}, \underbrace{\iota^* J}_{\text{almost complex}}) \overset{\iota}{\hookrightarrow} (M, J)$

Note: If $u : \Sigma \to M$ is J-hol., then so is $u \circ \varphi$ for

$$Aut(\Sigma, j) = \{ \varphi : \Sigma \to \Sigma \ \ (j,j)\text{-holomorphic diffeomorphism} \}.$$

$$\left(d(u \circ \varphi) \circ j = du \circ d\varphi \circ j = du \circ j \circ d\varphi = J \circ du \circ d\varphi = J \circ d(u \circ \varphi) \right)$$

Homework
Ex: $Aut(\underset{\mathbb{C} \cup \{\infty\}}{\overset{\shortparallel}{\mathbb{CP}^1}}, i) = \left\{ z \mapsto \frac{az+b}{cz+d} \ \middle| \ \begin{matrix} a,b,c,d \in \mathbb{C} \\ ad - bc = 1 \end{matrix} \right\}$ Möbius transformations $= PSL(2, \mathbb{C})$

We will study moduli spaces of "pseudoholomorphic curves" C resp. $[u]$ for fixed $A \in H_2(M)$, $p_0 \in M$

$$\{ C \subset M \ \text{J-hol. submfd}, \ C \simeq \mathbb{CP}^1, \ [C] = A, \ p_0 \in C \}$$

$\updownarrow$ 1-1 if all u embedded

$$\mathcal{M}(J, A) := \{ u : \mathbb{CP}^1 \to M \mid \bar{\partial}_J u = 0, \ u_*[\mathbb{CP}^1] = A, \ u(\infty) = p_0 \} \big/ Aut(\mathbb{CP}^1, i, \infty)$$

$$Aut(\mathbb{CP}^1, i, \infty) = \{ \varphi(\infty) = \infty \} = \{ z \mapsto az + b \}$$

RECALL:

Gromov nonsqueezing: If $B^{2n}(R) \xrightarrow{symp} B^2(1)\times\mathbb{C}^{n-1}$ then $R \le 1$

$\boxed{\text{Plan of Proof}}$ We will prove $R \le r \quad \forall r > 1$ by

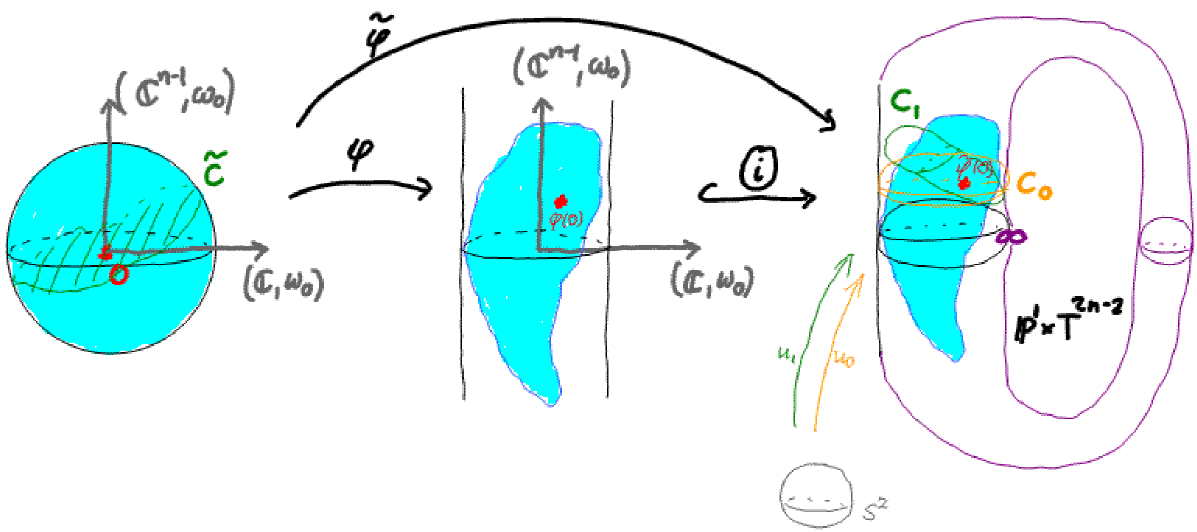
(i) embed $\overset{\mathbb{C}\supset}{B(1)} \xhookrightarrow{symp} (\mathbb{CP}^1, r\omega_{std})$ $\int_{\mathbb{CP}^1} \omega_{std} = \pi$

$pr_{\mathbb{C}^{n-1}}(im\,\varphi) \xhookrightarrow{symp} \left(T^{2n-2} = \mathbb{R}^{2n-2}/\mathbb{Z}^{2n-2}, K\omega_0\right)$ $\int_{T^{2n-2}} \omega_0\wedge\dots\wedge\omega_0 = 1$

to get $\tilde\varphi : B^{2n}(R) \hookrightarrow \left(\mathbb{CP}^1\times T^{2n-2}, r\omega_{std}\times K\omega_0\right)$

(ii) find a $\tilde\varphi_* J_0$-holomorphic curve $C_1 \subset \mathbb{CP}^1\times T^{2n-2}$ through $\tilde\varphi(0)$
(extended beyond $im\,\tilde\varphi$) $[C_1] = [\mathbb{CP}^1\times pt] \in H_2$

(iii) apply monotonicity lemma to $\tilde C = \tilde\varphi^{-1}(C_1)$



$\boxed{\text{Sketch of (ii)}}$

$$\mathcal{M}(J_0) := \{ C \subset \mathbb{P}^1 \times T \mid \overset{\sqrt{-1}\times\sqrt{-1}}{J_0}\, TC = TC ,\ \tilde{\varphi}(0) \in C ,\ [C] = [\mathbb{P}^1 \times pt] \}$$

consists of one curve , $C_0 = \mathbb{P}^1 \times pr_T(\tilde{\varphi}(0))$

View this as J_0-hol. map $u_0 : \mathbb{CP}^1 \to \mathbb{CP}^1 \times T^{2n-2}$, $z \mapsto (z, pr_T \tilde{\varphi}(0))$ satisfying $u_0(\underset{\overset{\|}{pr_{\mathbb{CP}^1}\tilde{\varphi}(0)}}{z_0}) = \tilde{\varphi}(0)$, $u_{0*}[\mathbb{CP}^1] = [\mathbb{CP}^1 \times pt]$

and note that $u_0 \circ \psi$ for $\psi : \mathbb{CP}^1 \to \mathbb{CP}^1$ diffeomorphism, i-holomorphic also parametrizes C_0, is also J_0-holomorphic

Claim : $\exists\ u_1 : \mathbb{CP}^1 \to \mathbb{CP}^1 \times T$, $\bar{\partial}_{J_1} u_1 = 0$, $u_1(z_0) = \tilde{\varphi}(0)$, $u_{1*}[\mathbb{CP}^1] = [\mathbb{CP}^1 \times pt]$

(a) $$\mathcal{M}(J_0) := \{ u : \mathbb{CP}^1 \to \mathbb{CP}^1 \times T^{2n-2} \mid \bar{\partial}_{J_0} u = 0,\ u(z_0) = \tilde{\varphi}(0),\ u_*[\mathbb{CP}^1] = [\mathbb{CP}^1 \times pt] \} / \mathrm{Aut}(\mathbb{CP}^1, i, z_0)$$
$$= \{[u_0]\}$$ $\supseteq$ ⟶ homework problem (using energy identity)

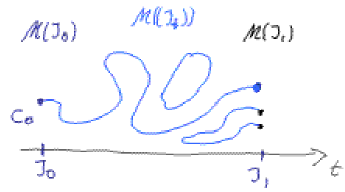
(b) find "$\underset{\text{TBD}}{\text{regular}}$" continuous family $[0,1] \to \mathcal{J}(\mathbb{CP}^1 \times T^{2n-2})$, $t \mapsto J_t$

from J_0 to J_1 = extension of $\varphi_* \sqrt{-1}$

(c) $$\mathcal{M}((J_t)) := \{ (t \in [0,1],\ C \subset \mathbb{P}^1 \times T) \mid J_t \cdot TC = TC ,\ \tilde{\varphi}(0) \in C ,\ [C] = [\mathbb{P}^1 \times pt] \}$$
$$= \{ (t, u) \mid \bar{\partial}_{J_t} u = 0 , \ldots \} / u \circ \psi \sim u$$

is

- a manifold — "transversality"/"regularization"
- dimension 1 — "Fredholm theory"
- compact — "Gromov compactness"
- has boundary $\partial \mathcal{M}((J_t)) = \mathcal{M}(J_0) \cup \mathcal{M}(J_1)$ — "bubbling analysis"



$\Rightarrow \mathcal{M}(J_1) \neq \emptyset$ since compact 1-manifolds have even number of boundary points