L3 - The Cauchy-Riemann equation

Remark: J-holomorphic maps/submanifolds are "very rare" unless

- ⊛ M is complex ⇝ algebraic geometry:

  (locally: holomorphic functions $F: M \to \mathbb{C}^r$ cut out holomorphic submanifolds $F^{-1}(0) \subset M$ of dimension $\dim M - 2r$ if/where dF is surjective)

OR

- ⊛ $\dim_{\mathbb{R}} \Sigma = 2$ ⇝ $\bar\partial_J$ is a nonlinear elliptic PDE

Pseudoholomorphic curves ($\dim_{\mathbb{R}} \Sigma = 2$) behave almost like holomorphic functions

- **Carleman similarity principle**

$u : \mathbb{C} \to \mathbb{R}^{2n}$, $J, C : \mathbb{C} \to \mathbb{R}^{2n \times 2n}$, $J^2 = -\mathbb{1}$

$$\partial_s u + J \partial_t u + C u = 0 \quad , \quad u(0) = 0$$

Rmk: $\bar\partial_J u = 0 \iff$ in local coord. $\partial_s u + J \partial_t u = 0$

$\Longrightarrow \exists \, \delta > 0$, $\Phi : B_\delta(0) \to \mathbb{R}^{2n \times 2n}$ invertible:

$$\Phi^{-1} J \Phi = J_0 \quad , \quad \Phi^{-1} u =: v \text{ is holomorphic} : \partial_s v + J_0 \partial_t v = 0$$

This does not prove regularity ($\bar\partial_J u = 0 \Rightarrow u \in \mathcal{C}^\infty$) or estimates (like $\|u\|_{W^{1,2}} \le \underset{\text{independant of } u}{C} (\|\bar\partial_J u\|_{L^2} + \|u\|_{L^2})$)

since
- u must map to Darboux chart
- Φ is as regular as $J = J \circ u$

But it does prove other useful properties of J-holomorphic curves:

- **Unique continuation**: Σ connected, $u, v : \Sigma \to M$ J-hol.

  $B \subset \Sigma$ open, $u|_B = v|_B$ (or $u = v$ to ∞ order at a point)

  $\Rightarrow$ $u \equiv v$

Cor.: $u : \Sigma \to M$ J-hol., not constant, Σ compact

- $\Rightarrow u^{-1}(p)$ finite $\forall p \in M$
- $\Rightarrow \mathrm{Crit}\, u = \{ z \in \Sigma \mid du(z) = 0 \}$ finite

- **Thm**: $u : \Sigma \to M$ J-holomorphic, simple$^{\circledast}$

  ($\circledast$ i.e. there is no $\varphi : (\Sigma, j) \to (\Sigma', j')$, $\deg \varphi > 1$ such that $\underbrace{u = v \circ \varphi}_{\text{"multiply covered"}}$

$$\boxed{\begin{array}{l} \varphi_*[\Sigma] = \deg\varphi \cdot [\Sigma'] \\ \text{OR } \deg\varphi = \underbrace{\#\varphi^{-1}(z_0)}_{\text{count with sign}} \end{array}}$$

$$\begin{tikzcd} \Sigma \arrow[r, "u"] \arrow[d, "\varphi"'] & M \\ \Sigma' \arrow[ur, "v"'] & \end{tikzcd}$$

)

$\Rightarrow \underbrace{\{ z \in \Sigma \mid du(z) \neq 0,\ u^{-1}(u(z)) = \{z\} \}}_{\text{"injective points"}} \subset \Sigma$ open, dense

(crucial for regularity of $\mathcal{M}(A, J)$)

Example: $u : \mathbb{CP}^1 \to \mathbb{CP}^1 \times T$, $u_*[\mathbb{CP}^1] = [\mathbb{CP}^1 \times pt]$

$u = v \circ \varphi$, $\quad \mathbb{CP}^1 \times T \xrightarrow{pr} \mathbb{CP}^1$, $\quad \Downarrow$

$$\deg(pr_{\mathbb{CP}^1} \circ u) = 1$$

$$\deg(pr_{\mathbb{CP}^1} \circ v \circ \varphi) = \underbrace{\deg(pr_{\mathbb{CP}^1} \circ v)}_{\in \mathbb{Z}} \cdot \deg(\varphi) = 1 \quad \Rightarrow \deg(\varphi) = 1$$

$\Rightarrow$ u simple

$\boxed{\text{Cauchy-Riemann operator}}$

(Σ, j) Riemann surface

(M, J) almost complex manifold

$u : \Sigma \to M$

$\rightsquigarrow \big(d_z u : T_z\Sigma \to T_{u(z)}M\big)_{z\in\Sigma} \quad \rightsquigarrow \quad du : T\Sigma \to u^*TM = \big(T_{u(z)}M\big)_{z\in\Sigma}$

$$\begin{array}{ccc} T\Sigma & \longrightarrow & u^*TM \\ \downarrow & & \downarrow \\ \Sigma & \longrightarrow & \Sigma \end{array}$$

$$\Rightarrow du \in \Omega^1(\Sigma; u^*TM) = \{ \eta : \underset{\circlearrowleft j}{T\Sigma} \to \underset{\circlearrowleft J(u)}{u^*TM} \mid \eta(T_z\Sigma) \subset T_{u(z)}M \}$$

$$= \Omega^{1,0} \oplus \Omega^{0,1}$$

$$\Omega^{1,0} = \{\eta \circ j = J\circ\eta\}, \qquad \Omega^{0,1} = \{\eta\circ j = -J\circ\eta\}$$

<u>Def$^{\underline{n}}$</u>: $\bar\partial_J u := \frac{1}{2}(du + J\,du\,j)$ is the projection of du to $\Omega^{0,1}$

<u>Note</u>: • $\bar\partial_J$ is nonlinear in u: $\bar\partial_J u\,(X\in T_z\Sigma) = \frac{1}{2}\big(du(X) + J(u(z))\,du(j(z)X)\big)$

• in local coordinates $u : (\underset{s+it}{\overset{\Psi}{\mathbb{C}}}, i) \longmapsto (\mathbb{R}^{2n}, J : \mathbb{R}^{2n} \to \mathbb{R}^{2n\times 2n})$

$$\bar\partial_J u(\partial_s) = J\cdot \bar\partial_J(\partial_t) = \partial_s u(s,t) + J(u(s,t))\partial_t u(s,t)$$

• linearized operator $D_u\bar\partial_J \;:\; \xi \longmapsto \partial_s\xi + J(u)\partial_t\xi$
(TBD) is a linear Cauchy-Riemann operator $\Rightarrow$ elliptic

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(weak) compactness properties of $\mathcal{M}(J,A)$ hinge on

<u>energy identity</u> for ω-compatible J

$$E(u) := \frac{1}{2}\int_\Sigma \underbrace{|du|^2 dvol_\Sigma} = \int_\Sigma \underbrace{|\bar\partial_J u|^2 dvol_\Sigma} + \int_\Sigma u^*\omega$$

in local coordinates $|\eta|^2 dvol_\Sigma = \big(|\eta(\partial_s)|^2 + |\eta(\partial_t)|^2\big)\, ds\wedge dt = \langle \eta \wedge \overset{= -\eta\circ j}{\widetilde{*\eta}} \rangle_{g_J}$

$\Rightarrow$ this expression only depends on ω, J, j, not the metric on Σ

(g_Σ is determined by j "up to conformal rescaling" $g_\Sigma \rightsquigarrow f\cdot g_\Sigma$; $f\in\mathcal{C}^\infty(\Sigma,(0,\infty))$)

Proof: $4|\bar\partial_J u|^2 dvol = |du + J du\, j|^2 dvol$ $\qquad j\partial_s = \partial_t$, $j\partial_t = -\partial_s$

$= (|\partial_s u + J\partial_t u|^2 + |\partial_t u - J\partial_s u|^2)\, ds\wedge dt$

$= (|\partial_s u|^2 + |J\partial_s u|^2 + |J\partial_t u|^2 + |\partial_t u|^2 + 2g(\partial_s u, J\partial_t u) - 2g(\partial_t u, J\partial_s u))\, ds\wedge dt$

$= 2|du|^2 + 4\underbrace{\omega(\partial_s u, J^2\partial_t u)\, ds\wedge dt}_{= -u^*\omega}$ ∎

Note: $\bar\partial_J u = 0$ & u embedding $\Rightarrow E(u) = \frac{1}{2}\int |du|^2 dvol_\Sigma = \text{area}_{g_J} \text{im}\, u$

" locally

$\left(\int \frac{1}{2}(|\partial_s u|^2 + |\partial_t u|^2\, ds\,dt = \int |\partial_s u|\cdot|\partial_t u|\, ds\,dt \ ;\ \partial_s u = -J\partial_t u \perp \partial_t u\right)$

Cor.: • J-hol. curves of fixed homology $u_*[\Sigma] = A$ have fixed energy

$$E(u) = \tfrac{1}{2}\|du\|^2_{L^2} = \int_\Sigma u^*\omega = \langle A, [\omega]\rangle$$

• null-homologous J-hol. curves are constant $\left(\int |du|^2 = 0\right)$

• J-hol. curves minimize energy $\Rightarrow$ harmonic maps ("Δu = lower order")

" $\bar\partial_J$ is a 1st order reduction of Δ "

locally: $\partial_s u + J(u)\partial_t u = 0$

$(\nabla_s - J\nabla_t)$ $\Longrightarrow$ $\underbrace{\Delta u}_{\text{Laplace operator}} = \underbrace{(\nabla_{\partial_t u} J)\partial_s u - (\nabla_{\partial_s u} J)\partial_t u}_{\text{nonlinear but lower order}}$

(Ex) J-hol. curves minimize area $\Rightarrow$ minimal surface

Def$^{\underline{n}}$: $C \subset (M, g)$ 2-dim. submanifold is <u>minimal</u> if for all continuous families $(C_t)_{t\in(-\varepsilon,\varepsilon)}$ with $C_0 = C$ and $\exists K \subset M$ compact: $\forall t$ $C_t \setminus K = C_0 \setminus K$ the area $(C_t) = \int_{C_t} dvol_{g|_{C_t}}$ has local minimum at $t = 0$