Distributions

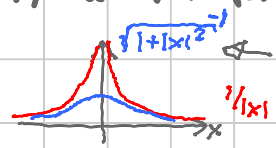
Notation

$$\mathcal{C}_0^k(\mathbb{R}^n) := \left\{ f:\mathbb{R}^n\to\mathbb{C} \;\middle|\; k\text{-fold continuously differentiable},\ \sup_{|x|>R} |D^\alpha f(x)| \xrightarrow[R\to\infty]{} 0 \quad \forall |\alpha|\le k \right\}$$

is a Banach space with norm

$$\|f\|_{\mathcal{C}^k} := \sup_{|\alpha|\le k}\|D_\alpha f\|_\infty \;,\; \|f\|_\infty = \sup_{x\in\mathbb{R}^n}|f(x)| \,,\; D_\alpha = (-i)^{|\alpha|}\left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} \quad \text{for } \alpha=(\alpha_1,\dots,\alpha_n),\ |\alpha| = \alpha_1+\dots+\alpha_n$$

Schwartz space ("test functions")



← could replace this by any smooth function asymptotic to $1/|x|$

$$\mathcal{S}(\mathbb{R}^n) := \left\{ \varphi:\mathbb{R}^n\to\mathbb{C} \;\middle|\; \sqrt{1+|x|^2}^{\,k}\varphi \in \mathcal{C}_0^k \quad \forall k\in\mathbb{N}_0 \right\} = \bigcap_{k\in\mathbb{N}_0} \overbrace{\sqrt{1+|x|^2}^{\,-k}\,\mathcal{C}_0^k}^{X_k}$$

is a metric space with $d(\varphi,\psi) := \sum 2^{-k}\dfrac{d_k}{1+d_k} = \displaystyle\sum_{k=0}^{\infty} 2^{-k}\frac{\|\sqrt{1+|x|^2}^{\,k}(\varphi-\psi)\|_{\mathcal{C}^k}}{1+\|\sqrt{1+|x|^2}^{\,k}(\varphi-\psi)\|_{\mathcal{C}^k}}$

Ex: $\mathcal{C}_c^\infty \subset \mathcal{S}(\mathbb{R}^n)$ (compact support), $e^{-|x|^2}\in\mathcal{S}(\mathbb{R}^n)\setminus\mathcal{C}_c^\infty$, $\dfrac{1}{\sqrt{1-|x|^2}}\in\mathcal{C}_0^k\setminus\mathcal{S}\quad\forall k$

Lemma (i) $\|\varphi\|_{X_k} = \|\sqrt{1+|x|^2}^{\,k}\varphi\|_{\mathcal{C}^k}$ equivalent to $\displaystyle\sup_{|\alpha|,|\beta|\le k}\|x^\alpha D_\beta\varphi\|_\infty =: \|\varphi\|_{k,\infty}$

i.e. $\frac{1}{C}\|\cdot\|_{X_k} \le \|\cdot\|_{k,\infty} \le C\|\cdot\|_{X_k}$

(ii) $\varphi_i \xrightarrow[i\to\infty]{} \varphi$ in $(\mathcal{S}(\mathbb{R}^n),d)$ $\iff$ $\forall k\in\mathbb{N}_0\quad \varphi_i\xrightarrow[i\to\infty]{}\varphi$ in (X_k,d_k)

(iii) $T:\mathcal{S}(\mathbb{R}^n)\longrightarrow\mathcal{S}(\mathbb{R}^n)$ linear

T continuous $\iff \forall k\ \exists N,C\ \forall\varphi\in\mathcal{S}(\mathbb{R}^n)$: $\displaystyle\sup_{|\alpha|,|\beta|\le k}\|\underbrace{x^\alpha}_{=x_1^{\alpha_1}\cdots x_n^{\alpha_n}} D_\beta(T\varphi)\|_\infty \le C\sup_{|\alpha'|,|\beta'|\le N}\|x^{\alpha'}D_{\beta'}\varphi\|_\infty$

Ex: $\varphi\mapsto x^\alpha\varphi := x_1^{\alpha_1}\cdots x_n^{\alpha_n}\varphi$, $\quad\varphi\mapsto D_\alpha\varphi$ are continuous

Proof of (ii) "$\Rightarrow$" since $\dfrac{d_k(\varphi_i,\varphi)}{1+d_k(\varphi_i,\varphi)} \le 2^k d(\varphi_i,\varphi)\to 0 \implies d_k(\varphi_i,\varphi)\to 0$

"$\Leftarrow$" given $\varepsilon>0$, first choose N so that $\displaystyle\sum_{k=N}^{\infty} 2^{-k}\frac{d_k(\varphi_i,\varphi)}{1+d_k(\varphi_i,\varphi)} \le \sum_{k=N}^{\infty}2^{-k} = 2^{-N+1} < \varepsilon/2$

then choose I so that for $k<N$: $d_k(\varphi_i,\varphi) < 2^k\dfrac{\varepsilon}{2N}\quad\forall i\ge I \implies \displaystyle\sum_{k=0}^{N-1}2^{-k}\frac{d_k(\varphi_i,\varphi)}{1+d_k(\varphi_i,\varphi)} < \frac{\varepsilon}{2}$

<u>tempered distributions</u> : $\mathcal{S}'(\mathbb{R}^n) :=$ dual space of $\mathcal{S}(\mathbb{R}^n) \overset{⊛}{=} \bigcup_{k\in\mathbb{N}} X_k^*$

$= \{ u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ linear, continuous $\}$

with weak topology

$$u_i \xrightarrow[i\to\infty]{\mathcal{S}'} u \;:\Leftrightarrow\; \forall \varphi \in \mathcal{S} \quad u_i(\varphi) \xrightarrow{\mathbb{C}} u_\infty(\varphi)$$

⊛ $u \in \mathcal{S}' \Rightarrow \exists k : u \in X_k^*$ holds because of continuity (esp. $\exists \varepsilon > 0 : d(\varphi, 0) < \varepsilon \Rightarrow |u(\varphi)| < 1$),
the estimate $d(\varphi, 0) \le \left(\sum_{i=0}^{k} 2^{-i}\right)\|\varphi\|_{X_k} + \sum_{i=k+1}^{\infty} 2^{-i} \le 2\|\varphi\|_{X_k} + 2^{-k}$
(and hence $\exists k \in \mathbb{N}, \delta > 0 : \|\varphi\|_{X_k} \le \delta \Rightarrow |u(\varphi)| < 1$)
and linearity ($\Rightarrow \forall \psi \in \mathcal{S}\setminus\{0\} : |u(\psi)| = |u(\delta \cdot \psi / \|\psi\|_{X_k})| \cdot \delta^{-1}\|\psi\|_{X_k} \le \delta^{-1}\|\psi\|_{X_k}$

<u>Ex</u>: Delta distribution at $p \in \mathbb{R}^n$: $\delta_p : \varphi \mapsto \varphi(p)$

<u>Lemma</u>: $f \mapsto u_f(\varphi) := \int_{\mathbb{R}^n} f \cdot \varphi \, d^n x$ is a continous dense injection $\mathcal{S}(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n)$

and extends to continuous injections $L^p(\mathbb{R}^n) \hookrightarrow \mathcal{S}'(\mathbb{R}^n) \quad \forall\, 1 \le p \le \infty$

<u>Proof</u>: $|u_f(\varphi)| = |\int f \cdot \varphi| \le \|f\|_{L^p} \cdot \|\varphi\|_{L^q} \le C \|f\|_{L^p} \|\varphi\|_{X_k}$ (Hölder inequality with $\frac{1}{p} + \frac{1}{q} = 1$)

proves $u_f \in \mathcal{S}'$ and continuity wrt $f \in L^p$

because

• $p > 1, q < \infty$: $\|\varphi\|_{L^q}^q = \int |\varphi|^q d^n x = \int_0^\infty \int_{S^{n-1}} \left| \sqrt{1+|x|^2}^{\,k} \cdot \varphi \right|^q \sqrt{1+r^2}^{\,-kq} r^{n-1} d\mathrm{vol}_{S^{n-1}} dr$

$$\le \|\varphi\|_{X_k}^q \cdot \mathrm{Vol}_{S^{n-1}} \cdot \underbrace{\int_0^\infty \frac{r^{n-1}}{\sqrt{1+r^2}^{\,kq}} dr}_{\sim r^{n-1-kq} \text{ for } r \gg 1}$$

$\Rightarrow$ converges for $n - kq > 0$ ($\Leftrightarrow k > \frac{n}{q}$)

• $p = 1, q = \infty$: $\|\varphi\|_{L^q} = \|\varphi\|_{L^\infty} = \|\varphi\|_{X_0}$

Injectivity ($f \neq 0 \Rightarrow \exists \varphi \in \mathcal{S} : u_f(\varphi) \neq 0$):

• For $f \in \mathcal{S}$, $f(x) \neq 0$ can use $\varphi = f \cdot \psi$ with ψ cutoff function supported in $\{x \mid f(x) \neq 0\}$
to get $u_f(\varphi) = \int f \cdot f\psi = \int |f|^2 \cdot \psi > 0$.

- For $f \in L^p$: $f \neq 0 \in L^p \underset{\lightning}{\Rightarrow} u_f \neq 0 \in (L^q)^* \overset{\mathcal{C}_c^\infty \subset L^q \text{ dense}}{\Longrightarrow} \exists \varphi \in \mathcal{C}_c^\infty : u_f(\varphi) \neq 0$

($L^p \hookrightarrow (L^q)^*$ is isometric embedding for $\frac{1}{p} + \frac{1}{q} = 1$ (just not surjective for $p=1$))

Density of $\mathcal{S} \hookrightarrow \mathcal{S}'$ can be proven later (by convolution). For now, we will use continuous extension as guiding principle to define **operations on $\mathcal{S}'(\mathbb{R}^n)$**:

$x^\alpha \cdot u\,(\varphi) := u(x^\alpha \varphi)$ $\qquad D_\alpha u\,(\varphi) := u((-1)^{|\alpha|} D_\alpha \varphi)$

$\psi \cdot u\,(\varphi) := u(\psi \varphi)$ $\qquad$ for $\psi \in \mathcal{S}(\mathbb{R}^n)$

(Note e.g. that $u_{D_\alpha f}(\varphi) = \int D_\alpha f \cdot \varphi = \int (-1)^{|\alpha|} f \cdot D_\alpha \varphi = u_f((-1)^{|\alpha|} D_\alpha \varphi)$ $\forall f, \varphi \in \mathcal{S}$ and $[\varphi \mapsto u((-1)^{|\alpha|} D_\alpha \varphi)] \in \mathcal{S}'$ $\forall u \in \mathcal{S}'$. So this is the continuous extension of $f \mapsto D_\alpha f$ on $\mathcal{S}$.)

Fourier transform: $\quad \mathcal{F}: \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$

$$\varphi \mapsto \hat{\varphi}(\xi) = \int e^{-ix\cdot\xi} \varphi(x)\, d^n x$$

is continuous by Lemma since $\widehat{x_j \varphi} = -D_{\xi_j} \hat{\varphi}$, $\widehat{D_{x_j} \varphi} = \xi_j \hat{\varphi}$

* inversion: $(2\pi)^n f(x) = \hat{\hat{f}}(-x)$

* Parseval: $(2\pi)^{-n/2} \mathcal{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is a Hilbert space isomorphism

in particular $u_f(\hat{\varphi}) = \underset{\iint f(x) e^{ix\cdot y} \varphi(y)\, d^n x\, d^n y}{\int f \cdot \hat{\varphi} = \int \hat{f} \cdot \varphi} = u_{\hat{f}}(\varphi)$ $\qquad \forall f, \varphi \in \mathcal{S}$

Def$^{\underline{n}}$: $\mathcal{F} : \mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n) \qquad u \mapsto \hat{u}(\varphi) := u(\hat{\varphi})$

This will be important because of the convolution property: $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$ which yields symbolic solution of PDE's : $\sum c_\alpha D_\alpha u = f$ $\qquad u = \mathcal{F}^{-1}\left(\frac{1}{\sum c_\alpha \xi^\alpha}\right) * f$

$\Updownarrow$ in $\mathcal{S}'$ $\qquad\qquad \Updownarrow$

$\sum c_\alpha \xi^\alpha \hat{u} = \hat{f} \quad \Rightarrow \quad \hat{u} = \frac{1}{\sum c_\alpha \xi^\alpha} \hat{f}$

Ex (important for fundamental solutions of PDEs): $\hat{\delta} = 1$ (i.e. $= u_1$)

$$\hat{\delta}(\varphi) = \delta(\hat{\varphi}) = \hat{\varphi}(0) = \int e^{ix\cdot 0} \varphi(x) d^n x = \int 1 \cdot \varphi = u_1(\varphi)$$

We can also use Fourier transform to define Sobolev spaces on $\mathbb{R}^n$:

$$H^s(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \sqrt{1+|\xi|^2}^{\,s} \hat{u} \in L^2(\mathbb{R}^n) \} \quad \text{for any } s \in \mathbb{R}$$

Preview: $H^k(\mathbb{R}^n) = W^{k,2}(\mathbb{R}^n)$ for $k \in \mathbb{N}_0$, where

$$W^{k,p}(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid D_\alpha u \in L^p(\mathbb{R}^n) \ \forall |\alpha| \le k \} \quad \text{for } k \in \mathbb{N},\ 1 \le p \le \infty$$

is the more general definition of Sobolev spaces (applies to domains other than $\mathbb{R}^n$)

* These are all Banach spaces with norms $\|u\|_{H^s} = \| \sqrt{1+|\xi|^2}^{\,s} \hat{u} \|_{L^2}$

$$\|u\|_{W^{k,p}} = \max_{|\alpha| \le k} \|D_\alpha u\|_{L^p} \quad \text{or} \quad \Big(\sum_{|\alpha| \le k} \|D_\alpha u\|_{L^p}^p \Big)^{1/p}$$

all norms of type are equivalent $\quad u \longmapsto (\|D_\alpha u\|_{L^p})_{|\alpha| \le k} \in \mathbb{R}^N \xrightarrow{\text{norm on } \mathbb{R}^n} \mathbb{R}$

* $W^{k,p}(\mathbb{R}^n) \subset \mathcal{C}^\ell_0(\mathbb{R}^n)$ for $k \gg \ell$ (more precisely: Sobolev embedding)

$$\Rightarrow \bigcap_{k \in \mathbb{N}} W^{k,p}(\mathbb{R}^n) \subset \mathcal{C}^\infty_0(\mathbb{R}^n)$$

* $\mathcal{S}(\mathbb{R}^n) \subset W^{k,p}(\mathbb{R}^n) \quad \forall k,p$ (by estimates as for $\mathcal{S} \hookrightarrow \mathcal{S}'$)

Similar to derivatives and multiplication on $\mathcal{S}'$, we define

support / **singular support** of distributions

to agree as best possible with the notion on $\mathcal{S}$:

$$\operatorname{supp} u := \mathbb{R}^n \setminus \{x \in \mathbb{R}^n \mid \exists \psi \in \mathcal{S}(\mathbb{R}^n) : \psi(x) \neq 0 \,,\ \psi \cdot u = 0\}$$

$$\operatorname{singsupp} u := \mathbb{R}^n \setminus \{x \in \mathbb{R}^n \mid \exists \psi \in \mathcal{S}(\mathbb{R}^n) : \psi(x) \neq 0 \,,\ \psi \cdot u \in \mathcal{S}(\mathbb{R}^n)\}$$

$$= \qquad \mathcal{C}_c^\infty \qquad \mathcal{C}_c^\infty$$

Lemma: (i) $\operatorname{supp} u_f = \overline{\{x \in \mathbb{R}^n \mid f(x) \neq 0\}} \qquad \forall f \in \mathcal{S}$

(ii) $\operatorname{supp} u \subset \mathbb{R}^n$ is closed $\qquad \forall u \in \mathcal{S}'$

(ii)' $\operatorname{singsupp} u \subset \mathbb{R}^n$ is closed $\qquad \forall u \in \mathcal{S}'$

(iii) $\operatorname{supp} u = \{y \in \mathbb{R}^n \mid \nexists$ open neighbourhood $V \subset \mathbb{R}^n$ of y s.t. $u|_V = 0\}$

i.e. $u(\varphi) = 0 \ \ \forall \varphi \in \mathcal{S}, \operatorname{supp} \varphi \subset V$

(iii)' $\operatorname{singsupp} u = \{y \in \mathbb{R}^n \mid \nexists$ open neighbourhood $V \subset \mathbb{R}^n$ of y s.t. $u|_V \in \mathcal{C}^\infty\}$

i.e. $\exists f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$: $u = u_f$ on $\{u \in \mathcal{S} \mid \operatorname{supp} \varphi \subset V\}$

Proof by writing out what things mean & hint for (ii): if $\psi(x) \neq 0$ then $\psi(x') \neq 0$ for x' in neighbourhood of x

Example: $\operatorname{supp} \delta_p = \operatorname{singsupp} \delta_p = \{p\}$

$\operatorname{supp} D_\alpha \delta_p = \operatorname{singsupp} D_\alpha \delta_p = \{p\} \qquad \forall \alpha$

Thm: * $\operatorname{supp} u = \emptyset \iff u = 0$

* $\mu \in \mathcal{C}_c^\infty$, $\operatorname{supp}\mu \cap \operatorname{singsupp} u = \emptyset \Rightarrow \mu \cdot u \in \mathcal{C}_c^\infty$ (i.e. $\mu \cdot u = u_f$ for some $f \in \mathcal{C}_c^\infty$)

* $K \subset \mathbb{R}^n$ compact, $K \cap \operatorname{singsupp} u = \emptyset \Rightarrow \exists \mu \in \mathcal{C}_c^\infty : \mu|_K \equiv 1$, $\mu \cdot u \in \mathcal{C}_c^\infty$

* $u \in \mathcal{S}'$, $x_j \cdot u = 0 \ \forall j = 1..n \iff u = c \cdot \delta_0$ for some $c \in \mathbb{C}$

* $u \in \mathcal{S}'$, $\operatorname{supp} u = \{p\} \iff u = \sum_{|\alpha| \leq k} c_\alpha D_\alpha \delta_p$ for some $k \in \mathbb{N}$, $c_\alpha \in \mathbb{C}$

not so hard via Fourier

"$\Rightarrow$" more complicated but interesting to know that "point support characterizes Dirac delta & its derivatives"

$\boxed{\text{Convolution}}$ Def$^{\underline{n}}$: $(f*g)(x) = \int f(y)g(x-y)\,d^ny$ for e.g. $f \in L^1, g \in \mathcal{S}$

- For $f \in \mathcal{S}(\mathbb{R}^n)$, $\begin{array}{l} L^p(\mathbb{R}^n) \longrightarrow W^{k,p}(\mathbb{R}^n) \\ g \longmapsto f*g \end{array}$ is continuous for $1 \le p \le \infty$, $k \in \mathbb{N}$

$\Big[$ since $D_\alpha(f*g) = (D_\alpha f)*g$, $\|f*g\|_{L^p} \le \|f\|_{L^1} \cdot \|g\|_{L^p}$ (Young ineq.) $\Big]$

- For $f, g \in \mathcal{S}$ $\widehat{f*g} = \hat{f} \cdot \hat{g}$ $\Rightarrow$ $(2\pi)^n\, \widehat{f \cdot g} = \hat{f} * \hat{g}$

- $\mathcal{S} * \mathcal{S} \subset \mathcal{S}$ since $f, g \in \mathcal{S} \Rightarrow f*g \in \bigcap_{k \in \mathbb{N}} W^{k,p} \subset \mathcal{C}_0^\infty$

decay estimate: $\left|\int f(y)g(x-y)\,d^ny\right| \le \left|\int \sqrt{1+|y|^2}^{\,k} f(y) \sqrt{1+|x-y|^2}^{\,k} g(x-y)\,d^ny\right|$

$\cdot \sup_{y \in \mathbb{R}^n} \sqrt{(1+|y|^2)(1+|x-y|^2)}^{\,-k}$

$\left(\begin{array}{l} |x|^2 \le (|y|+|x-y|)^2 \le 4\max(|y|,|x-y|)^2 \\ \quad \le 4(|y|^2+|x-y|^2) \end{array}\right)$

$$\le \sqrt{\tfrac{1}{4}(1+|x|^2)}^{\,-k} \,\|\sqrt{1+|y|^2}^{\,k} f\|_{L^2} \,\|\sqrt{1+|y|^2}^{\,k} g\|_{L^2}$$

$$\le C \sqrt{1+|x|^2}^{\,-k} \|f\|_{X^\ell} \|g\|_{X^\ell} \quad \text{for } \ell > n+k$$

To extend convolution to $\mathcal{S}'$, note that $(f*\varphi)(x) = u_f(\varphi_x)$ for $\varphi_x(y) := \varphi(x-y)$

Hence it makes sense to define

Def$^{\underline{n}}$: $(u*\varphi)(x) := u(\varphi_x)$ for $u \in \mathcal{S}'$, $\varphi \in \mathcal{S}$.

Note: $u*\varphi \in \mathcal{C}^\infty(\mathbb{R}^n)$ (but not necc. in $\mathcal{S}$) since $\mathbb{R}^n \to \mathcal{S}$, $x \mapsto \varphi_x$ is continuous

and $\partial_{x_i}(u*\varphi)((x_1..x_n)) = \lim_{h\to 0} \frac{1}{h}\left(u(\varphi_{(x_1..x_i+h..x_n)}) - u(\varphi_{(x_1..x_n)})\right)$

$$= \lim_{h\to 0} u\Big(\underbrace{y \mapsto \tfrac{1}{h}\big(\varphi((x_1..x_i+h..x_n)-y) - \varphi((x_1..x_n)-y)\big)}_{\text{converges in } \mathcal{S} \text{ as function of } y \;\xrightarrow[h\to 0]{}\; (\partial_{x_i}\varphi)(x-y)}\Big)$$

To extend further, we need another algebraic property: associativity.

Lemma: $u \in \mathcal{S}'$, $v, \varphi \in \mathcal{S}$ $\Rightarrow$ $(u * v) * \varphi = u * (v * \varphi)$

$(\Rightarrow v * \varphi \in \mathcal{S})$

For $u, v \in \mathcal{S}'$ we would like to define $u * v$ by this, that is

$$(u * v)(\varphi) \overset{\text{def}}{=} ((u * v) * \varphi_0)(0) \overset{!?!}{=} (u * (v * \varphi_0))(0) \overset{\text{def}}{=} u((v * \varphi_0)_0) \qquad \forall \varphi \in \mathcal{S}$$

$\varphi = (\varphi_0)_0$ since $\varphi_0(z) = \varphi(-z)$

Caution: this does not even define $u * v$ as an element of $(\mathcal{C}_c^\infty)^*$ since

$$v \in \mathcal{S}',\ \varphi \in \mathcal{C}_c^\infty \not\Rightarrow v * \varphi_0 \in \mathcal{S} \qquad \underline{\text{Ex}}: v = 1,\ \int \varphi = 1 \Rightarrow v * \varphi_0 = 1$$

So it's time for more estimates:

Recall: $\mathcal{S}' = \bigcup_{k \in \mathbb{N}} X_k^*$; $X_k = \sqrt{1+|x|^2}^{\,k}\, \mathcal{C}_0^k$ so $u \in \mathcal{S}' \Leftrightarrow u \in X_k^*$ for some k

Lemma: (i) $u \in X_k^*, \varphi \in \mathcal{S} \Rightarrow |(u * \varphi)(x)| \leq C \sqrt{1+|x|^2}^{\,k} \|\varphi\|_{X_k}$

(ii) $u \in X_k^*$, supp u compact $\Rightarrow \|u * \varphi\|_{X_\ell} \leq C_\ell \|u\|_{X_k^*} \|\varphi\|_{X_{k+\ell}} \quad \forall \varphi \in \mathcal{S}, \ell \in \mathbb{N}$

Corollary (i) $u \in \mathcal{S}', \varphi \in \mathcal{S} \Rightarrow u * \varphi \in \mathcal{S}' \cap \mathcal{C}^\infty$

(ii) $u \in \mathcal{S}_c' = \{u \in \mathcal{S}' \mid \text{supp } u \text{ compact}\} \Rightarrow \varphi \mapsto u * \varphi$ is continuous $\mathcal{S} \to \mathcal{S}$

(iii) $\mathcal{S} \hookrightarrow \mathcal{S}'$, $f \mapsto u_f$ is dense

$\Rightarrow |\varphi(0) - \int \psi_i \varphi| \leq \int_{\text{supp } \psi_i} \psi_i(y) \cdot |\varphi(0) - \varphi(y)|\, d^n y \leq \sup_{|y| \leq 2^{-i}} |\varphi(0) - \varphi(y)| \cdot \int \psi_i \leq 2^{-i} \|D\varphi\|_\infty$

e.g. supp $\psi \subset B_1(0)$

Take $\psi \in \mathcal{C}_c^\infty$, $\int \psi = 1$ then $\psi_i(y) := 2^{-ni} \cdot \psi(2^i y)$ is a Dirac sequence: $u_{\psi_i} \xrightarrow[i \to \infty]{X_1^*} \delta$

Now for $u \in X_k^*$, $u * \psi_i \xrightarrow{\mathcal{S}'} u$ since $(u * \psi_i)(\varphi) = u((\psi_i * \varphi_0)_0) \xrightarrow{X_k} u((\varphi_0)_0) = u(\varphi) \quad \forall \varphi$

$$\left(\|\psi_i * \varphi_0 - \varphi_0\|_{X_k} \overset{(ii)}{\leq} C \|\psi_i - \delta\|_{X_1^*} \|\varphi_0\|_{X_{k+1}} \right)$$

Proof: (i) $|(u*\varphi)(x)| = |u(\varphi_x)| \le \|u\|_{X_k^*}\, \|\varphi(x-\cdot)\|_{X_k} \le \|u\| \cdot C \|\varphi(x-\cdot)\|_{k,\infty}$

$$\left[\|\varphi(x-\cdot)\|_{k,\infty} = \sup_{|\alpha|,|\beta|\le k} \sup_y \left| y^\beta \left(\frac{\partial}{\partial y}\right)^\alpha \varphi(x-y) \right| = \sup_{\alpha,\beta} \sup_z \Big| \underbrace{(x-z)^\beta}_{\sum_{\beta_1+\beta_2=\beta} c_{\beta_1\beta_2} x^{\beta_1} z^{\beta_2}} D_\alpha \varphi(z) \Big| \right.$$

$$\left. \le C_k (1+|x|^k) \sup_{|\alpha|,|\beta_1|\le k} \sup_z |z^{\beta_1} D_\alpha \varphi(z)| \le C_k' \sqrt{1+|x|^2}^{\,k}\, \|\varphi\|_{X_k} \right]$$

(ii) pick $\mu \in C_c^\infty$, $\mu|_{\operatorname{supp} u} = 1$, $\operatorname{supp}\mu \subset B_R$ $\Rightarrow \mu \cdot u = u$

$$\Rightarrow (u*\varphi)(x) = (\mu\cdot u)(\varphi_x) = u(y \mapsto \mu(y)\varphi(x-y))$$

$$\|u*\varphi\|_{X_\ell} \le C \sup_{|\alpha|,|\beta|\le \ell} \|x^\beta D_\alpha (u*\varphi)\|_\infty$$

$$= C \sup_{\alpha,\beta} \sup_x \left| x^\beta\, u(y \mapsto \mu(y)(D_\alpha\varphi)(x-y)) \right|$$

$$\le C \sup_{|\alpha|,|\beta|\le \ell} \sup_x \|u\|_{X_k^*} \underbrace{\|y \mapsto x^\beta \mu(y)(D_\alpha\varphi)(x-y)\|_{X_k}}_{\le C \sup_{|\alpha'|,|\beta'|\le k} \sup_y |x^\beta y^{\beta'} D_{\alpha'}(\underbrace{\mu(y)}_{\equiv 0 \text{ for } |y|>R} D_\alpha \varphi(x-y))|}$$

$(x = z+y)$

$$\le C' \|u\|_{X_k^*} \sup_{\substack{|\alpha|,|\beta|\le \ell \\ |\alpha'|,|\beta'|\le k}} \sup_{\substack{z \\ |y|\le R}} \underbrace{|(z+y)^\beta y^{\beta'} D_{\alpha'+\alpha}\varphi(z)|}_{\le C \sup_{\beta=\gamma+\gamma'} |y^{\gamma+\beta'}|\, |z^{\gamma'} D_{\alpha'+\alpha}\varphi(z)|}$$

$$\le C'' R^{k+\ell} \|u\|_{X_k^*} \|\varphi\|_{X_{k+\ell}} \qquad \blacksquare$$

For $u, v \in \mathcal{S}'$ with supp v compact we can now define $u * v$ by

$$(u * v)(\varphi) := u(\underbrace{(v * \varphi_0)_0}_{\in \mathcal{S}}) \qquad \forall \varphi \in \mathcal{S}$$

Def^{n}: For $u, v \in \mathcal{S}'$, with one having compact support define $u * v \in \mathcal{S}'$ by

$$(u * v)(\varphi) := \begin{cases} u((v * \varphi_0)_0) & \text{if } v \in \mathcal{S}'_c \\ u(\mu \cdot (v * \varphi_0)_0) & \text{if } u \in \mathcal{S}'_c,\ \mu \in \mathcal{C}^\infty_c,\ \mu|_{\text{supp}\, u} = 1 \end{cases}$$

HW:

- This is well defined ($u * v \in \mathcal{S}'$ and doesn't depend on choice of e.g. μ)
- $D^\alpha (u * v) = (D^\alpha u) * v = u * (D^\alpha v)$
- supp $(u * v) \subset$ supp(u) + supp(v)

partial differential operator with constant coefficients:

$$p(D) = \sum_{|\alpha|\leq m} p_\alpha D_\alpha \qquad \text{for } m\in\mathbb{N} \text{ "order"},\ p_\alpha \in \mathbb{C} \text{ "coefficients"}$$

defines continuous linear maps $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$.

The <u>symbol</u> is the polynomial $p(\xi) = \sum_{|\alpha|\leq m} p_\alpha \xi_\alpha \ : \mathbb{R}^n \to \mathbb{C}$

The leading order symbol $p_m(\xi) := \sum_{|\alpha|=m} p_\alpha \xi_\alpha$ gives a partial classification:

$p(D)$ is

- <u>elliptic</u> if $p_m(\xi)=0 \Rightarrow \xi = 0$
- <u>hyperbolic</u> on $\mathbb{R}\times\mathbb{R}^n$ if $\forall \xi \in \mathbb{R}^n\setminus\{0\}$ $p_m(\xi_0,\xi)=0$ has m distinct real solutions
- <u>parabolic</u> on $\mathbb{R}\times\mathbb{R}^n$ if $p(D) = \partial_{x_0} + \overset{\text{real valued}}{\text{elliptic}}$ in $x_1 \dots x_n$

<u>Examples</u>:

		lead symbol	
ell.	Laplacian operator	$\xi_1^2 + \dots + \xi_n^2$	$\Delta = -\left(\frac{\partial}{\partial x_1}\right)^2 - \dots - \left(\frac{\partial}{\partial x_n}\right)^2$
ell.	Cauchy-Riemann operator	$-i\xi_1 + \xi_2$	$\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2}$
hyp.	Wave operator	$-\xi_0^2 + (\xi_1^2 + \dots + \xi_n^2)$	$\left(\frac{\partial}{\partial x_0}\right)^2 \underbrace{-\left(\frac{\partial}{\partial x_1}\right)^2 - \dots - \left(\frac{\partial}{\partial x_n}\right)^2}_{\Delta}$
par.	Heat operator	$\xi_1^2 + \dots + \xi_n^2$	$\frac{\partial}{\partial x_0} + \Delta$
none	Schrödinger operator	$\xi_1^2 + \dots + \xi_n^2$	$-i\frac{\partial}{\partial x_0} + \Delta$ or $\frac{\partial}{\partial x_0} + i\Delta$

<u>Rmk</u>: If you saw ellipticity for varying coefficients on bundles, then note that

$p(D)$ elliptic $\Leftrightarrow$ $\mathbb{C} \to \mathbb{C}$, $z \mapsto p_m(\xi) z$ is an isomorphism $\forall \xi \neq 0$

Def$^{\underline{n}}$: $F \in \mathcal{S}'(\mathbb{R}^n)$ is a fundamental solution of $p(D)$ if $p(D)F = \delta$

Thm: Every $p(D) \not\equiv 0$ has a fundamental solution.
!hard!

Symbolically, $F = \mathcal{F}^{-1}\left(1/p(\xi)\right)$ is a fundamental solution

(since $p(D)F = \delta \overset{\mathcal{F}}{\Leftrightarrow} p(\xi)\hat{F} = 1$) but usually $\frac{1}{p(\xi)} \notin \mathcal{S}'$.

If F is fundamental solution with compact support, then

$u \mapsto F * u$ is inverse to $p(D): \mathcal{S}' \to \mathcal{S}'$.
$\hat{u} \mapsto \hat{F} \cdot \hat{u}$ $\qquad$ $\hat{u} \mapsto p(\xi)\hat{u}$

But most $p(D)$ are not bijective $\Rightarrow$ compactly supported F.S. are rare

Def$^{\underline{n}}$: $F \in \mathcal{S}'(\mathbb{R}^n)$ is a parametrix of $p(D)$ if $p(D)F - \delta \in \mathcal{C}^\infty(\mathbb{R}^n)$

$p(D)$ is <u>hypoelliptic</u> if it has a parametrix F with $\operatorname{singsupp} F = \{0\}$

Thm 1: Elliptic operators are hypoelliptic. (The Heat operator is parabolic and hypoelliptic.)

Thm 2: $p(D)$ hypoelliptic $\Rightarrow$ $\operatorname{singsupp}(p(D)u) \underset{\overset{\subset}{\text{always}}}{=} \operatorname{singsupp} u \quad \forall u \in \mathcal{S}'$

Cor ("elliptic regularity"): $p(D)u \in \mathcal{C}^\infty(\mathbb{R}^n) \Rightarrow u \in \mathcal{C}^\infty(\mathbb{R}^n)$

singsupp $= \emptyset$

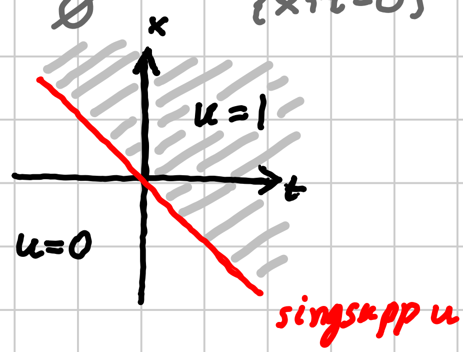
Ex: $u \in \mathcal{S}'$, $\bar{\partial} u = 0 \Rightarrow u \in \mathcal{C}^\infty$

Counterexample to "regularity" and "$\operatorname{singsupp}(p(D)u) \supset \operatorname{singsupp} u$"

$\operatorname{singsupp}(p(D)u) = \emptyset$, $\operatorname{singsupp} u = \{x+t=0\}$

$$\underbrace{(\partial_t^2 - \partial_x^2)}_{p(D)} \underbrace{H(x+t)}_{u} = 0 \qquad ; H(y) = \begin{cases} 1 & ; y>0 \\ 0 & ; y\le 0 \end{cases}$$

Wave operator



<u>Recall</u>: $\operatorname{singsupp} u = \mathbb{R}^n \setminus \{x\in\mathbb{R}^n \mid \exists \varphi \in \mathcal{C}_c^\infty : \varphi(x)\neq 0, \ \varphi\cdot u \in \mathcal{C}_c^\infty\}$

closed

$$= \{x\in\mathbb{R}^n \mid \nexists\, \mathcal{V} \text{ open nbhd of } x : \underset{\sim}{u|_{\mathcal{V}} \in \mathcal{C}_c^\infty}\}$$

i.e. $\exists f\in\mathcal{C}_c^\infty : \forall \varphi\in\mathcal{S} \ \operatorname{supp}\varphi\subset\mathcal{V} : u(\varphi) = \int f\cdot\varphi$

<u>Lemma</u>: If $P(D)$ has a parametrix F with $\operatorname{singsupp} F$ compact, (resp. $\operatorname{singsupp} F = \{0\}$)

then it also has a parametrix $\tilde F \in \mathcal{S}'$ with $\operatorname{supp} \tilde F$ compact, (resp. $\operatorname{supp} F \subset B_\varepsilon(0)$ for any given $\varepsilon>0$)

and $P(D)\tilde F - \delta = \tilde\psi \in \mathcal{C}_c^\infty$. (resp. $\operatorname{supp}\tilde\psi \subset B_\varepsilon(0)$)

<u>Sketch</u>: Pick $\mu\in\mathcal{C}_c^\infty$, $\mu|_{\operatorname{singsupp} F} \equiv 1$ then $\tilde F := \mu\cdot F \in \mathcal{S}_c'$
(resp. $\mu(0)=1$, $\operatorname{supp}\mu\subset B_\varepsilon(0)$)

$$P(D)(\mu\cdot F) = \mu\cdot \underbrace{P(D)F}_{\delta+\psi} + \underbrace{\sum_{\substack{1\le|\alpha|\le m \\ |\beta|\le m-|\alpha|}} c_{\alpha\beta}\, \underset{\sim}{D_\alpha\mu}\cdot D_\beta F}_{\mathcal{C}_c^\infty}$$

$c_{\alpha\beta} \in \mathbb{C}$, $D_\alpha\mu \equiv 0$ on $\operatorname{singsupp} F$

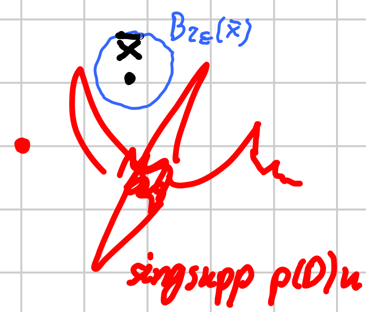
$$= \underbrace{\mu\cdot\delta}_{=\delta \text{ since } \mu(0)=1} + \underbrace{\mu\cdot\psi + \psi'}_{\tilde\psi\in\mathcal{C}_c^\infty}$$

■

Proof of Thm 2 $\left[\operatorname{singsupp}(p(D)u) \supset \operatorname{singsupp} u\right]$

F parametrix with $\operatorname{singsupp} F = \{0\}$; w.l.o.g. $\operatorname{supp} F \subset B_\varepsilon(0)$ (by Lemma)

Given $u \in \mathcal{S}'$, $\quad \bar{x} \notin \underbrace{\operatorname{singsupp} P(D)u}_{\text{closed in } \mathbb{R}^n}$



need to find $\varphi \in \mathcal{C}_c^\infty$, $\varphi(\bar{x}) = 1$, $\varphi \cdot u \in \mathcal{C}_c^\infty$

pick $\varepsilon > 0$ s.t. $B_{2\varepsilon}(\bar{x}) \cap \operatorname{singsupp} P(D)u = \emptyset$

$$h \in \mathcal{C}_c^\infty, \; h|_{B_{2\varepsilon}(\bar{x})} \equiv 1, \; \operatorname{supp} h \cap \operatorname{singsupp} p(D)u = \emptyset$$

$$u = \delta * u = (P(D)F) * u - \underbrace{\psi * u}_{\mathcal{C}^\infty} \qquad ; \psi \in \mathcal{C}_c^\infty$$

$$= (P(D)u) * F - \psi * u$$

$$= \underbrace{(h \cdot P(D)u) * F}_{\mathcal{C}_c^\infty} + \underbrace{((1-h) \cdot P(D)u) * F}_{\operatorname{supp}(..) \subset \bar{B}_{2\varepsilon}(\bar{x})^c + B_\varepsilon(0) \subset \mathbb{R}^{2n} \setminus B_\varepsilon(\bar{x})} - \psi * u$$

For $\varphi \in \mathcal{C}_c^\infty$, $\varphi(\bar{x}) = 1$, $\operatorname{supp} \varphi \subset B_\varepsilon(\bar{x})$

$$\varphi \cdot u = \varphi \cdot (h \cdot P(D)u) * F - \varphi \cdot \psi * u \in \mathcal{C}_c^\infty \qquad \blacksquare$$

Lemma:

(a) $F \in \mathcal{S}'$ fundamental solution $\Rightarrow$ $\forall f \in \mathcal{S} \cup \mathcal{S}'_c : p(D)(F * f) = f$

(but "$p(D)u = f \Rightarrow u = (p(D)F) * u = F * f$" requires $u \in \mathcal{S} \cup \mathcal{S}'_c$)

(b) $F \in \mathcal{S}'_c$ parametrix, $p(D)F - \delta = \psi \in \mathcal{C}^\infty_c$

$$\Rightarrow \boxed{u = \delta * u = (p(D)F - \psi) * u = F * (p(D)u) - \psi * u}$$

Note: $(F*) \circ p(D) = Id + K_\psi$, $K_\psi u = \psi * u$

If $\|K_\psi\| < 1$ in some operator norm, then $p(D)$ has left inverse $\sum_{\ell=0}^{\infty} (-K_\psi)^\ell \circ (F*)$.

Proof of Thm 1 $\quad p(D) = p_m(D) + \text{lower order}, \quad p_m^{-1}(0) \subset \{0\}$

Step 0: $\exists R, \delta > 0 : \forall |\xi| \geq R \quad |p(\xi)| \geq \delta |\xi|^m$

$$|(D_\alpha \tfrac{1}{p})(\xi)| \leq C_\alpha |\xi|^{-m-|\alpha|}$$

Sketch: * $|p_m(\xi)| \geq \underbrace{\max_{|\xi|=1} \frac{|p_m(\xi)|^m}{|\xi|^m}}_{"\delta_m > 0} \cdot |\xi|^m$ since p_m homogeneous

$\delta_m > 0$ since $p_m|_{|\xi|=1} \neq 0$

* pick R s.t. $\underbrace{|p(\xi) - p_m(\xi)|}_{\text{order } m-1 \text{ polynomial}} \leq \frac{1}{2}\delta_m |\xi|^m \quad \forall |\xi| \geq R$

* $\left|D_i \frac{1}{p}\right| = \left|\frac{-D_i p}{p^2}\right| \leq \underbrace{|D_i p|}_{\text{order } m-1} \cdot \frac{1}{|p|^2} \leq C |\xi|^{m-1} \cdot (\delta |\xi|^m)^{-2}$

$$\leq C_i |\xi|^{-m-1}$$

* similarly for higher derivatives

Claim: $F := \mathcal{F}^{-1}\left(\frac{1-h}{p}\right)$ is a parametrix for $h \in \mathcal{C}_c^\infty$, $h|_{B_R(0)} \equiv 1$

- $\frac{1-h}{p} \in \mathcal{C}^\infty$, $\frac{1-h}{p} \le \delta^{-1}|\xi|^{-m}$ $\Rightarrow \frac{1-h}{p} \in \mathcal{S}' \Rightarrow F \in \mathcal{S}'$
- $p(D)F = \mathcal{F}^{-1}\left(p\cdot\frac{1-h}{p}\right) = \mathcal{F}^{-1}(1-h) = \delta - \underbrace{\mathcal{F}^{-1}(h)}_{\in \mathcal{S}}$
- singsupp $F \subset \{0\}$: Given $x \neq 0$ find $\varphi \in \mathcal{C}_c^\infty : \varphi(x) \neq 0,\ \varphi\cdot F \in \mathcal{C}_c^\infty$

Choose φ so that $\varphi|_{\mathcal{U}} \equiv 0$ for a neighbourhood $\mathcal{U}$ of 0. Then

$$\varphi\cdot F = \frac{\varphi}{|x|^{2\ell}} \cdot \underbrace{|x|^{2\ell} F}_{\in \mathcal{C}_0^k \text{ for } 2\ell > k+n-m} \in \mathcal{C}_c^\infty \cdot \mathcal{C}_0^k \subset \mathcal{C}_c^k$$

Find such ℓ for all k $\Rightarrow \varphi F \in \mathcal{C}_c^\infty$

$$\left|\sqrt{1+|\xi|^2}^{\,s} D_\alpha \frac{1-h}{p}\right| \le C\,|\xi|^{s-m-|\alpha|}$$

$\left[\sqrt{1+|\xi|^2} \le \sqrt{2}\,|\xi| \text{ on } B_R^c\ ;\ R>1\right]$

$\Rightarrow$ ———— $\in L^2$ for $s < m + |\alpha| - \frac{n}{2}$

$\left[\int (r^\lambda)^2 r^{n-1}\,dr < \infty \text{ for } 2\lambda + n - 1 < -1\right]$

$\Rightarrow x^\alpha F \in H^s$ for $s < m + |\alpha| - n/2$

$\left[H^s \hookrightarrow \mathcal{C}_0^0 \text{ for } s > n/2\right]$

$\Rightarrow x^\alpha F \in \mathcal{C}_0^k$ for $k < m + |\alpha| - n$

■ since $|x|^{2\ell} = \left(\sum_{i=1}^n x_i^2\right)^\ell$, $|\alpha| = 2\ell$

& a similar argument proves

Schwartz representation theorem: $u \in \mathcal{S}'(\mathbb{R}^n) \Rightarrow$

$\Rightarrow u \in \left(\sqrt{1+|x|^2}^{\,-k}\mathcal{C}_0^k\right)^*$ for some k $\Rightarrow \sqrt{1+|x|^2}^{\,-k} u \in (H^r)^*$ for $r > \frac{n}{2} + k$

Sobolev duality fact: $(H^r(\mathbb{R}^n))^* = H^{-r}(\mathbb{R}^n) = \left\{ \sum_{|\alpha| \le r+m} D_\alpha v_\alpha \;\middle|\; v_\alpha \in H^m(\mathbb{R}^n) \right\}$ for any $r, m \in \mathbb{N}_0$

$\left[\text{pick } m > \frac{n}{2}\right.$ $\hookrightarrow \mathcal{C}_0^0(\mathbb{R}^n)$

$$\Rightarrow u = \sqrt{1+|x|^2}^{\,k} \sum_{|\alpha| \le M} D_\alpha v_\alpha \quad (M = r+m)$$

$$= \ldots = \sum_{|\alpha| \le M} x^\alpha D_\beta \underbrace{u_{\alpha\beta}}_{\in \mathcal{C}_0^0} = \ldots = \sum_{|\alpha| \le M} D_\beta \big(x^\alpha \underbrace{v_{\alpha\beta}}_{\in \mathcal{C}_0^0}\big)$$

Thm 3 (elliptic regularity) $\quad p(D)$ elliptic of order m ; $1<p<\infty$

$$u \in L^p,\ p(D)u \in W^{k,p} \Rightarrow u \in W^{k+m,p},\quad \|u\|_{W^{k+m,p}} \leq C\left(\|p(D)u\|_{W^{k,p}} + \|u\|_{L^p}\right)$$

(depends on k,p)

Note : $u \in L^p$ is a necessary assumption since e.g. $u = 1 \notin L^p$ although $\Delta u = 0$

Proof of Thm 3, $p=2$ $\qquad u = F * (p(D)u) - \psi * u$

- $\psi \in \mathcal{S}$, $u \in \mathcal{S}' \Rightarrow \psi * u \in \mathcal{C}^\infty$
- $D_\alpha(\psi * u) = (D_\alpha \psi) * u \in L^1 * L^2 \subset L^2 \quad \forall \alpha$

$\Big\}\Rightarrow \psi * u \in H^s \ \forall s$

- $\sqrt{1+|\xi|^2}^{\,m}\, \hat{F} \in L^\infty$ by using specific F as in Thm 1

$$\Rightarrow \sqrt{1+|\xi|^2}^{\,k+m}\, \widehat{F * (p(D)u)} = \sqrt{1+|\xi|^2}^{\,m}\, \hat{F} \cdot \sqrt{1+|\xi|^2}^{\,k}\, \widehat{p(D)u} \in L^\infty \cdot L^2 \subset L^2$$

$$\Rightarrow F * (p(D)u) \in H^{k+m} \qquad \Rightarrow u \in H^{k+m}$$

- $\|u\|_{H^{k+m}} \leq \|F * (p(D)u)\|_{H^{k+m}} + \|\psi * u\|_{H^{k+m}}$

$$= \left\| \sqrt{1+|\xi|^2}^{\,k+m}\, \hat{F}\, \widehat{p(D)u} \right\|_{L^2} + \sum_{|\alpha| \leq k+m} \|(D_\alpha \psi) * u\|_{L^2}$$

$$\leq \left\| \sqrt{1+|\xi|^2}^{\,m}\, \hat{F} \right\|_\infty \|p(D)u\|_{H^k} + \left(\sum_{|\alpha| \leq k+m} \|D_\alpha \psi\|_{L^1} \right) \|u\|_{L^2} \qquad \blacksquare$$

$\boxed{L^p\text{-multiplier}}$: [Stein, Harmonic Analysis / 18.155]

$T_\mu \varphi = \mathcal{F}^{-1}(\mu) * \varphi$

$\Updownarrow$

$\mu \in \mathcal{S}'(\mathbb{R}^n)$ defines $T_\mu : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$ by $\widehat{T_\mu \varphi} = \mu \cdot \hat{\varphi}$

Thm [Mihlin, Stein] $\| \xi^\alpha D_\alpha \mu \|_\infty < \infty \quad \forall |\alpha| \leq n \quad , \quad 1 < p < \infty$

$\Rightarrow \quad T_\mu : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ continuous (i.e. $\|T_\mu \varphi\|_{L^p} \leq C \|\varphi\|_{L^p} \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$)

(In fact, it suffices to check $\| \xi_{i_1} \ldots \xi_{i_\ell} \frac{\partial \mu}{\partial \xi_{i_1} \ldots \partial \xi_{i_\ell}} \|_\infty < \infty \quad \forall\, 1 \leq i_1 < \ldots < i_\ell \leq n$.)

Proof of Thm 3 for general $1 < p < \infty$; w.l.o.g. $k = 0$

$u, p(D)u \in L^p$; $u = F * (p(D)u) - \psi * u$ where we use as in Thm 1

$F = \mathcal{F}^{-1}\left(\frac{1-h}{p}\right)$; $h|_{B_R} = 1$ $\quad \left(\Rightarrow \quad \left|\left(D_\alpha \frac{1-h}{p}\right)(\xi)\right| \leq C_\alpha |\xi|^{-m-|\alpha|} \right)$

$\Rightarrow \mu_\beta := \mathcal{F}(D_\beta F) = \xi^\beta \cdot \frac{1-h}{p}$ is an L^p-multiplier for $|\beta| \leq m$

$$\left[\begin{aligned} \left\| \xi^\alpha D_\alpha \left(\xi^\beta \frac{1-h}{p(\xi)} \right) \right\|_\infty &\leq \sup_{|\xi| > R} |\xi|^{|\alpha|} \sum_{\alpha = \alpha_1 + \alpha_2} |\xi|^{|\beta| - |\alpha_1|} \underbrace{\left| D_{\alpha_2} \frac{1-h}{p(\xi)} \right|}_{\leq C |\xi|^{-m-|\alpha_2|}} \\ &\leq \sup_{|\xi| > R} C |\xi|^{|\alpha| + |\beta| - |\alpha_1| - |\alpha_2| - m} = \sup_{|\xi| > R} C |\xi|^{|\beta| - m} < \infty \quad \forall \alpha \end{aligned} \right]$$

$$\Rightarrow D_\beta u = \underbrace{(D_\beta F) * (p(D)u)}_{= T_{\mu_\beta}(p(D)u)} - (D_\beta \psi) * u \in T_{\mu_\beta}(L^p) + \mathcal{C}_c^\infty * L^p$$

$\in L^p \quad \forall |\beta| \leq m$

$\Rightarrow u \in W^{m,p}$

$$\|u\|_{W^{m,p}} \simeq \sum_{|\beta| \leq m} \|D_\beta u\|_{L^p} \leq \sum_{|\beta| \leq m} \left(\|T_{\mu_\beta}\|_{L^p \to L^p} \|p(D)u\|_{L^p} + \|D_\beta \psi\|_{L^1} \|u\|_{L^p} \right)$$

$$\leq C \left(\|p(D)u\|_{L^p} + \|u\|_{L^p} \right)$$

Note: If we drop the assumption $u \in L^p$, we still get regularity on any compact set $K \subset \mathbb{R}^n$: $u|_K \in W^{k+m,p}$

$$\left[\text{For } h \in \mathcal{C}_c^\infty, h|_K \equiv 1 : \quad hu = h \cdot \big(\underbrace{F * (p(D)u)}_{W^{k+m,p}}\big) - h \cdot (\underbrace{\psi * u}_{\mathcal{S} * \mathcal{S}' \subset \mathcal{C}^\infty}) \in W^{k+m,p} \right]$$

and an estimate for any compact $\Omega \subset \mathbb{R}^n$, $K \subset \operatorname{int}(\Omega)$

$$\|u\|_{W^{k+m,p}(K)} \leq C \left(\|p(D)u\|_{W^{k,p}(\Omega)} + \|u\|_{L^p(\Omega)} \right)$$

$$\|hu\|_{W^{k+m,p}} \leq \|h\|_{\mathcal{C}^{k+m}} \|F * (p(D)u)\|_{W^{k+m,p}(\operatorname{supp} h)} + \sum_{|\alpha|+|\beta| \leq k+m} \|D_\alpha h \cdot (D_\beta \psi) * u\|_{L^p}$$

w.l.o.g. $k=0$

Given K, Ω pick $h \in \mathcal{C}_c^\infty$, $h|_K \equiv 1$, $\operatorname{supp} h + B_\varepsilon(0) \subset \Omega$ for some $\varepsilon > 0$, and multiply parametrix with cutoff s.t. $\operatorname{supp} F, \operatorname{supp} \psi \subset B_\varepsilon(0)$

- $F * (p(D)u)|_{\operatorname{supp} h} = F * (\tilde{h} \cdot p(D)u)|_{\operatorname{supp} h}$ for $\tilde{h} \in \mathcal{C}_c^\infty$, $\tilde{h}|_{\operatorname{supp} h + B_\varepsilon} \equiv 1$, $\operatorname{supp} \tilde{h} \subset \Omega$

since $\forall \varphi \in \mathcal{S}$, $\operatorname{supp} \varphi \subset \operatorname{supp} h$ $\big(F * (\tilde{h} \cdot p(D)u)\big)(\varphi) = \int \underbrace{\tilde{h}(x)}_{\substack{=1 \\ \text{on } \operatorname{supp} h + B_\varepsilon}} \cdot p(D)u(x) \cdot \underbrace{F(\varphi_x)}_{\substack{=0 \text{ if } \varphi_x|_{B_\varepsilon(0)} = 0 \\ \text{i.e. } x \notin \operatorname{supp}\varphi + B_\varepsilon(0)}} \, d^n x$

$$\|F * (p(D)u)\|_{W^{m,p}(\operatorname{supp} h)} \leq \sum_{|\beta| \leq m} \|(D_\beta F) * (\tilde{h} \cdot p(D)u)\|_{L^p}$$

$$\leq \sum_{|\beta| \leq m} \|T_{\mu_\beta}\| \, \|\tilde{h} \cdot p(D)u\|_{L^p} \leq C \|p(D)u\|_{L^p(\Omega)}$$

- $\|D_\alpha h \cdot (D_\beta \psi) * u\|_{L^p} = \sup_{\|\varphi\|_{L^{p'}}=1} |u(\underbrace{((D_\beta \psi) * (D_\alpha h \cdot \varphi)_0)_0}_{\text{supported in } (\operatorname{supp}\psi + (\operatorname{supp} h)_0)_0})|$

$= \operatorname{supp} h + (\operatorname{supp} \psi)_0$ (reflection at 0)

$\subset \Omega$

by choice of h; $\operatorname{supp} \psi \subset B_\varepsilon(0)$

$$\leq \|u\|_{L^p(\Omega)} \sup_{\|\varphi\|_{L^{p'}}=1} \|(D_\beta \psi) * (D_\alpha h \cdot \varphi)_0\|_{L^{p'}}$$

$$\leq \|u\|_{L^p(\Omega)} \|D_\beta \psi\|_{L^1} \|D_\alpha h\|_{L^\infty}$$