Sobolev spaces on $\mathbb{R}^n$

Def$^{\underline{n}}$/Lemma: For $m \in \mathbb{N}_0$
$$H^m(\mathbb{R}^n) := \left\{ g \in L^2(\mathbb{R}^n) \;\middle|\; \sqrt{1+|x|^2}^{\,m}\, \hat{g} \in L^2(\mathbb{R}^n) \right\}$$
$$= \left\{ g \in L^2(\mathbb{R}^n) \;\middle|\; \underbrace{D^\alpha g \in L^2}\ \forall |\alpha| \le m \right\}$$

is a Hilbert space with $\left(\text{i.e. } D^\alpha g = u_{f_\alpha} \in \mathcal{S}' \text{ with } f_\alpha \in L^2\right)$

$$\langle g, h \rangle_{H^m} := \int_{\mathbb{R}^n} (1+|\xi|^2)^m\, \hat{g}(\xi)\, \overline{\hat{h}(\xi)}\, d^n\xi$$

$$\left(\text{or, equivalently, } \langle g, h \rangle_{L^{m,2}} := \sum_{|\alpha| \le m} \langle \partial_\alpha g, \partial_\alpha h \rangle_{L^2} \right)$$

<u>Proof</u>: $\|g\|_{H^m} = \|(1+|\xi|^2)^{m/2} \hat{g}\|_{L^2} = 0 \Leftrightarrow \hat{g} = 0 \Leftrightarrow g = 0$; Completeness below

Generalizations

- $H^m = L^{m,2} = W^{m,2}$ is well defined for $m \in \mathbb{R}$
- $L^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n) := \left\{ g \in L^p(\mathbb{R}^n) \;\middle|\; D^\alpha g \in L^p \ \forall |\alpha| \le m \right\}$

  for $m \in \mathbb{N}_0$, $1 \le p \le \infty$ is a Banach space with

$$\|g\|_{W^{m,p}} := \left(\sum_{|\alpha| \le m} \|\partial_\alpha g\|_{L^p}^p \right)^{1/p} \quad ; \; p < \infty$$
$$\max_{|\alpha| \le m} \|\partial_\alpha g\|_{L^\infty} \quad ; \; p = \infty$$

E.g. $L^p = L^{0,p} = W^{0,p}$ $(= H^0$ if $p=2)$

<u>Proof of completeness</u>: Identify $W^{m,p}(\mathbb{R}^n)$ with closed subset of $(L^p(\mathbb{R}^n))^N$

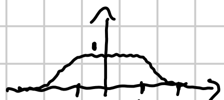
$$\Big\{ \underset{\substack{\shortparallel \\ (g_0,\, g_{(10\ldots0)},\, g_{(010\ldots0)},\, \ldots)}}{G = (g_\alpha)_{|\alpha| \le m}} \in \bigoplus_{|\alpha| \le m} L^p(\mathbb{R}^n) \;\Big|\; \underbrace{\int g_\alpha \cdot \varphi = - \int g_0 \cdot \partial_\alpha \varphi}_{\text{preserved under } L^p\text{-limit of } g_\alpha,\, g_0} \ \forall \varphi \in \mathcal{S} \ \forall |\alpha| \le m \Big\}$$

Def^n/Thm: For $m\in\mathbb{N}_0$, $1\le p<\infty$, $L^{m,p}(\mathbb{R}^n)=\overline{\mathcal{C}^\infty_c(\mathbb{R}^n)}^{\|\cdot\|_{L^{m,p}}}$,

i.e. $\mathcal{C}^\infty_c\subset L^{m,p}$ dense w.r.t. $\|g\|_{L^{m,p}}=\Big(\sum_{|\alpha|\le m}\|\partial_\alpha g\|^p_{L^p}\Big)^{1/p}$.

Step 1: $L^{m,p}_c(\mathbb{R}^n)\subset L^{m,p}(\mathbb{R}^n)$ is dense (not true for $p=\infty$)

$\quad = \{f\in L^{m,p}\mid \exists K\subset\mathbb{R}^n \text{ compact}: f\cdot\chi_{\mathbb{R}^n\setminus K}=0\in L^p\}$

Fix $\mu\in\mathcal{C}^\infty_c$  $\mu|_{B_1}\equiv 1$, $\operatorname{supp}\mu\subset B_2$, $\|\partial_\alpha\mu\|_\infty\le C_1\ \forall|\alpha|\le m$

For $g\in L^{m,p}$ have $\mu(t\,\cdot)g\in L^{m,p}_c\ \forall t>0$ ($\operatorname{supp}\subset B_{2t^{-1}}$) and

$$\|g-\underbrace{\mu(t\cdot)g}_{\operatorname{supp}\subset B_{2t^{-1}}}\|^p_{L^{m,p}}=\sum_{|\alpha|\le m}\|\partial_\alpha((1-\mu(t\cdot))g)\|^p_{L^p}$$

$$\le \|\underbrace{1-\mu(t\cdot)}_{=0\text{ on }B_{t^{-1}}}\|^p_{\mathcal{C}^m}\sum_{|\alpha|\le m}\int_{\mathbb{R}^n\setminus B_{t^{-1}}}|\partial_\alpha g|^p\le C_1^p\,\|g\|^p_{L^{m,p}(\mathbb{R}^n\setminus B_{t^{-1}})}\xrightarrow{t\to0}0$$

$$\|g\|^p_{L^{m,p}(\mathbb{R}^n\setminus B_{t^{-1}})}=\|g\|^p_{L^{m,p}(\mathbb{R}^n)}-\underbrace{\|g\|^p_{L^{m,p}(B_{t^{-1}})}}_{\xrightarrow{t\to0}\|g\|^p_{L^{m,p}(\mathbb{R}^n)}}$$

Step 2 $\mathcal{C}^\infty_c(\mathbb{R}^n)\subset L^{m,p}_c(\mathbb{R}^n)$ is dense (w.r.t. $\|\cdot\|_{L^{m,p}}$)

Let $\varphi_t(x)=t^{-n}\varphi(x/t)$ be a Dirac sequence with $\operatorname{supp}\varphi$ compact.

Given $f\in L^{m,p}_c$ we have $f*\varphi_t\in\mathcal{C}^\infty_c\ \forall t>0$, $f*\varphi_t\xrightarrow[t\to0]{L^p}f$, and

$$\forall|\alpha|\le m\quad \partial_\alpha(f*\varphi_t)=\partial^x_\alpha\int f(y)\varphi_t(x-y)\,d^ny\overset{\circledast}{=}\int f(y)\partial^x_\alpha\varphi_t(x-y)\,d^ny=\int f(y)(-1)^{|\alpha|}\partial^y_\alpha\varphi_t(x-y)\,d^ny$$

$\circledast$ (true for $f\in\mathcal{C}^\infty_c$ and preserved under L^p-limits)

$$=\int\underbrace{\partial_\alpha f}_{\in L^p}(y)\varphi_t(x-y)\,d^ny=(\partial_\alpha f)*\varphi_t\xrightarrow[t\to0]{L^p}\partial_\alpha f$$

Duality for Sobolev spaces

For $m \geq 0$ let $\quad H^{-m}(\mathbb{R}^n) := \{ u \in \mathcal{S}'(\mathbb{R}^n) \mid \sqrt{1+|\xi|^2}^{\,-m}\, \hat{u} \in L^2 \}$

$\left(\text{i.e. } \exists f \in L^2 : u\left(\sqrt{1+|\xi|^2}^{\,-m} \varphi\right) = \int f \cdot \varphi \quad \forall \varphi \in \mathcal{S} \right)$

<u>Prop$^{\underline{n}}$</u>: $\quad H^{-m}(\mathbb{R}^n) = \left\{ u = \sum_{|\alpha| \leq m} D_\alpha u_{f_\alpha} \;\middle|\; f_\alpha \in L^2(\mathbb{R}^n) \right\} = H^m(\mathbb{R}^n)^*$

Proof: • $\widehat{D_\alpha u_{f_\alpha}} = i^{|\alpha|} \xi^\alpha\, \widehat{u_{f_\alpha}} = i^{|\alpha|} \xi^\alpha u_{\hat{f}_\alpha} = u_{i^{|\alpha|} \xi^\alpha \hat{f}_\alpha}$

$$D_\alpha u_{f_\alpha} \in H^{-m} \quad \text{since} \quad \underbrace{\sqrt{1+|\xi|^2}^{\,-m}\; i^{|\alpha|} \xi^\alpha}_{|\cdot| \leq 1} \hat{f}_\alpha \in L^2 \quad \forall |\alpha| \leq m$$

• Given $u \in H^{-m}$ write

$$\hat{u} = \sum_{0 \leq |\alpha| \leq m} \xi^\alpha \underbrace{\operatorname{sign}(\xi^\alpha) \cdot \frac{\hat{u}}{\sum |\xi^\alpha|}}_{=: \hat{V}_\alpha \in L^2 \quad \text{since } \sum|\xi^\alpha| \geq c \sqrt{1+|\xi|^2}^{\,m}} = \widehat{\sum D_\alpha V_\alpha} \qquad \Rightarrow u = \sum D_\alpha V_\alpha$$

• $\underset{\mathcal{S}'}{\underset{\cap}{H^{-m}}} \subset (\underset{\mathcal{S}'}{\underset{\cap}{H^m}})^*$ since $\mathcal{S} \subset H^m$ $\qquad$ since $\quad u \in H^{-m}, \; g \in H^m$

$$\Rightarrow |u((2\pi)^n g(-\cdot))| = |u(\hat{\hat{g}})| = |\hat{u}(\hat{g})| = \left| \hat{u}\left(\sqrt{1+|\xi|^2}^{\,-m} \sqrt{1+|\xi|^2}^{\,m} \hat{g} \right) \right|$$

$$= \left| \left(\sqrt{1+|\xi|^2}^{\,-m} \hat{u} \right) \left(\sqrt{1+|\xi|^2}^{\,m} \hat{g} \right) \right| = \left| \int \left(\sqrt{1+|\xi|^2}^{\,-m} \hat{u} \right) \left(\sqrt{1+|\xi|^2}^{\,m} \hat{g} \right) \right|$$

$$\leq \left\| \sqrt{1+|\xi|^2}^{\,-m} \hat{u} \right\|_{L^2} \left\| \sqrt{1+|\xi|^2}^{\,m} \hat{g} \right\|_{L^2} = \|u\|_{H^{-m}} \|g\|_{H^m}$$

$$\Rightarrow |u(h)| \leq (2\pi)^{-n} \|u\|_{H^{-m}} \|h\|_{H^m} \qquad \forall h \in H^m$$

• $(H^m)^* \subset H^{-m}$ $\quad$ since $L \in (H^m)^* \Rightarrow \sqrt{1+|\xi|^2}^{\,-m/2} \hat{L} \in (L^2)^* \cong L^2$

$$\left(\left| \left(\sqrt{1+|\xi|^2}^{\,-m/2} \hat{L} \right)(\underset{L^2}{\underset{\cap}{h}}) \right| = \left| L\left(\widehat{\sqrt{1+|\xi|^2}^{\,-m/2} h} \right) \right| \leq \|L\|_{(H^m)^*} \left\| \widehat{\sqrt{1+|\xi|^2}^{\,-m/2} h} \right\|_{H^m} \right.$$

$$\left. = \|L\|_{(H^m)^*} \Big\| \underbrace{\sqrt{1+|\xi|^2}^{\,m/2}\; \widehat{\widehat{\sqrt{1+|\xi|^2}^{\,-m/2} h}}}_{(2\pi)^n h(-\xi)} \Big\|_{L^2} = (2\pi)^n \|L\|_{(H^m)^*} \|h\|_{L^2} \qquad \forall h \in L^2 \right)$$

Hölder spaces (other important function spaces) $\Omega \subset \mathbb{R}^n$

$\mathcal{C}^0(\Omega) = \{ g : \Omega \to \mathbb{C} \text{ continuous} : \forall x \in \Omega \ \forall \varepsilon > 0 \ \exists \delta(x,\varepsilon) > 0 : |y-x| < \delta \Rightarrow |g(y)-g(x)| < \varepsilon \}$

$g : \Omega \to \mathbb{C}$ is uniformly continuous if $\forall \varepsilon > 0 \ \exists \delta(\varepsilon) > 0 : |y-x| < \delta \Rightarrow |g(y)-g(x)| < \varepsilon$

— Lipschitz continuous if $\exists C' : |g(y)-g(x)| \leq C' |y-x|$

— Hölder continuous with exponent $0 < \lambda \leq 1$ if $\exists C' : |g(y)-g(x)| \leq C' |y-x|^\lambda$

(i.e. g uniformly continuous with $\delta = (C'^{-1}\varepsilon)^{1/\lambda}$)

— bounded if $\exists C : |g(x)| \leq C \ \forall x$

$\mathcal{C}^m_B(\Omega) = \{ g : \Omega \to \mathbb{C} \mid \forall |\alpha| \leq m \ \partial_\alpha g \text{ continuous, bounded} \}$ is a Banach space with

$$\|g\|_{\mathcal{C}^m(\Omega)} = \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |\partial_\alpha g(x)|$$

For $m \in \mathbb{N}_0$, $0 < \lambda \leq 1$

$\mathcal{C}^{m,\lambda}(\Omega) = \{ g : \Omega \to \mathbb{C} \mid \forall |\alpha| \leq m \ \partial_\alpha g \ \lambda\text{-Hölder continuous} \}$ is a Banach space with

$$\|g\|_{\mathcal{C}^{m,\lambda}(\Omega)} = \|g\|_{\mathcal{C}^m(\Omega)} + \sum_{|\alpha| \leq m} \sup_{x \neq y \in \Omega} \frac{|\partial_\alpha g(x) - \partial_\alpha g(y)|}{|x-y|^\lambda}$$

Arzela-Ascoli Thm (compactness) $\Omega \subset \mathbb{R}^n$ compact

$A \subset \mathcal{C}^0_B(\Omega)$ uniformly bounded: $\sup_{g \in A} \|g\|_{\mathcal{C}^0(\Omega)} < \infty$

equicontinuous: $\forall \varepsilon > 0 \ \exists \delta > 0 : g \in A, |x-y| < \delta$
$\Rightarrow |g(x)-g(y)| < \varepsilon$

$\Rightarrow A$ is precompact

Proof: [Lang, Real and Functional Analysis]

<u>Thm</u> (imbeddings)

(i) $\mathcal{C}^{m,\lambda}(\Omega) \hookrightarrow \mathcal{C}^{m,\mu}(\Omega) \hookrightarrow \mathcal{C}^{m}(\Omega)$ for $1 \geq \lambda > \mu > 0$

are continuous (and compact if Ω is compact)

(ii) $\mathcal{C}^{m+1}(\Omega) \hookrightarrow \mathcal{C}^{m,\lambda}(\Omega)$ for $1 \geq \lambda > 0$, Ω convex is continuous

<u>Proof</u>

(i) $$\sup_{0<|x-y|\leq 1} \frac{|\partial_\alpha g(x) - \partial_\alpha g(y)|}{|x-y|^\mu} \leq \sup_{x\neq y\in\Omega} \frac{|\partial_\alpha g(x) - \partial_\alpha g(y)|}{|x-y|^\lambda}$$

$$\sup_{|x-y|\geq 1} \frac{|\partial_\alpha g(x) - \partial_\alpha g(y)|}{|x-y|^\mu} \leq 2\|\partial_\alpha g\|_\infty$$

$\leq \|g\|_{\mathcal{C}^{m,\lambda}}$

Ω compact, $\|g_i\|_{\mathcal{C}^{m,\lambda}} \leq 1$ to find: $\mathcal{C}^{m}$- / $\mathcal{C}^{m,\mu}$- convergent subsequence

Arzela-Ascoli gives a subsequence s.t. $\partial_\alpha g_i \longrightarrow f_\alpha \in \mathcal{C}^0(\Omega)$ $\forall |\alpha| \leq m$

since $\sup_i \|\partial_\alpha g_i\|_\infty \leq 1$, $|\partial_\alpha g_i(x) - \partial_\alpha g_i(y)| \leq |x-y|^\lambda$

Let $f := f_{\underline{0}} = \lim g_i$, and check $f_\alpha = \partial_\alpha f$, $\|f - g_i\|_{\mathcal{C}^m} \to 0$ $\left(\begin{smallmatrix}\text{HW3}\\ \text{sol}^{\text{ns}}\end{smallmatrix}\right)$

$\Rightarrow (g_i)$ is $\mathcal{C}^m$-Cauchy $\Rightarrow$ $\mathcal{C}^{m,\mu}$-Cauchy since with $h = \partial_\alpha g_i - \partial_\alpha g_j$

$$\sup_{x\neq y} \frac{|h(x)-h(y)|}{|x-y|^\mu} \leq \left(\sup_{x\neq y} \frac{|h(x)-h(y)|}{|x-y|^\lambda}\right)^{\mu/\lambda} \cdot (2\|h\|_\infty)^{1-\mu/\lambda} \leq \underbrace{\left(\|g_i\|_{\mathcal{C}^{m,\lambda}} + \|g_j\|_{\mathcal{C}^{m,\lambda}}\right)}_{\text{bounded}}^{\mu/\lambda} \underbrace{(2\|g_i - g_j\|_{\mathcal{C}^m})}_{\to 0}^{1-\mu/\lambda}$$

$\Rightarrow g_i \xrightarrow[i\to\infty]{\mathcal{C}^{m,\mu}} f$ since $\mathcal{C}^{m,\mu}$ is complete

(ii) $$|\partial_\alpha g(x) - \partial_\alpha g(y)| \leq \int_0^1 |\partial_t \partial_\alpha g(\underbrace{y + t(x-y)}_{\in\Omega \text{ if convex}})|\,dt \leq \|g\|_{\mathcal{C}^{m+1}} |x-y|$$

$\Rightarrow \mathcal{C}^{m+1} \hookrightarrow \mathcal{C}^{m,1} \xrightarrow{(i)} \mathcal{C}^{m,\lambda}$ ∎

$\underline{\text{Def}^{\underline{n}}}$: X, Y normed spaces

- $X \subset Y$ <u>imbedding</u> if $X \subset Y$ subspace, $X \hookrightarrow Y$ continuous
- $i : X \to Y$ <u>compact</u> if $K \subset X$ bounded $\Rightarrow i(K) \subset Y$ precompact

i.e. $\sup_k \|x_k\|_X < \infty \Rightarrow \{i(x_k)\} \subset Y$ has convergent subseq.

$\Omega \subset \mathbb{R}^n$ with nonempty interior, smooth boundary, $m \in \mathbb{N}_0$, $1 \le p < \infty$

$$W^{m,p}(\Omega) := \{ f \in L^p(\Omega) \mid \forall |\alpha| \le m \ \ \partial_\alpha f \in L^p(\Omega) \} \overset{\text{Ext.}}{\underset{\text{Thm}}{=}} W^{m,p}(\mathbb{R}^n)|_\Omega$$

$f \in L^p(\Omega) \overset{\text{def}}{\Leftrightarrow} \tilde{f}(x) = \begin{cases} f(x), & x \in \Omega \\ 0, & x \notin \Omega \end{cases} \in L^p(\mathbb{R}^n)$

$\partial_\alpha f \in L^p(\Omega) \overset{\text{def}}{\Leftrightarrow} \exists g_\alpha \in L^p : \int f \cdot \partial_\alpha \varphi = \int g_\alpha \cdot \varphi \ \ \forall \varphi \in C_c^\infty(\Omega)$

(supp $\varphi \subset \Omega$)

$$\|f\|_{W^{m,p}(\Omega)} := \Big(\sum_{|\alpha| \le m} \int_\Omega |\underbrace{\partial_\alpha f}_{= g_\alpha}|^p \Big)^{1/p}$$

<u>Extension theorem</u>: If Ω "not wild" $\exists$ bounded linear operator $W^{m,p}(\Omega) \to W^{m,p}(\mathbb{R}^n)$ s.t. $E(f)|_\Omega = f$

for details see Adams, Sobolev spaces

For discussion of Sobolev spaces of
- maps between manifolds
- sections of vector bundles

see e.g. appendix of [Wehrheim, "Uhlenbeck Compactness"].

Sobolev imbedding theorem

$$W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,1-n/p}(\Omega) \quad ; \ p>n$$

$$W^{1,p}(\Omega) \hookrightarrow \bigcap_{q\geq p} L^q(\Omega) \quad ; \ p=n$$

(generally not L^∞ except for special case $W^{n,1}(\Omega\subset\mathbb{R}^n) \hookrightarrow \mathcal{C}^0_B(\Omega)$)

$$W^{1,p}(\Omega) \hookrightarrow L^{\frac{np}{n-p}}(\Omega) \quad ; \ p<n$$

Rellich thm: Ω compact

$$W^{1,p}(\Omega) \hookrightarrow L^r(\Omega) \quad \text{compact} \quad \forall r<\frac{np}{n-p} \ ; \ p<n, \qquad r<\infty \ ; \ p=n$$

$$W^{1,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\lambda}(\Omega) \quad \text{compact} \quad \forall\, 0<\lambda\leq 1-\frac{n}{p} \ ; \ p>n$$

$$\hookrightarrow \mathcal{C}^0(\Omega) \hookrightarrow L^r(\Omega) \quad \forall r$$

Corollary of Hölder & Sobolev imbedding

Corollary / general Sobolev imbedding

If $m>\frac{n}{p}>m-1$, $0<\lambda\leq m-\frac{n}{p}<1$ then

$$W^{m,p}(\Omega) \hookrightarrow \mathcal{C}^{0,\lambda}(\Omega) \hookrightarrow \mathcal{C}^0_B(\Omega)$$

$$W^{m+k,p}(\Omega) \hookrightarrow \mathcal{C}^{k,\lambda}(\Omega) \hookrightarrow \mathcal{C}^k_B(\Omega)$$

are continuous and compact if Ω compact, $\lambda<m-\frac{n}{p}$

If $mp\leq n$, $q\leq\frac{np}{n-mp}$, $q<\infty$, ($p\leq q$ or Ω compact) then

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega)$$

$$W^{m+k,p}(\Omega) \hookrightarrow W^{k,q}(\Omega)$$

are continuous and compact if Ω compact, $q<\frac{np}{n-mp}$

Proof of Corollary: • $g \in W^{m+k,p} \Leftrightarrow \partial_\alpha g \in W^{m,p} \quad \forall |\alpha| \le k$
$\qquad\qquad\qquad\quad\ \mathcal{C}^{m+k,\lambda} \qquad\qquad \mathcal{C}^{m,\lambda}$

• Hölder inequality $\Rightarrow$ $L^p(\Omega) \cap L^r(\Omega) \hookrightarrow L^q(\Omega) \quad \forall\, p \le q \le r$

$$L^r(\Omega) \hookrightarrow L^q(\Omega) \qquad \forall q \le r \text{ , } \Omega \text{ compact}$$

• iteration, e.g. $W^{2,p} \xhookrightarrow[p<n]{} W^{1,\frac{np}{n-p}} \xhookrightarrow[2p<n]{} L^{\frac{\cancel{n}\cdot\frac{np}{n-p}}{\cancel{n}-\frac{\cancel{n}p}{n-p}}}$ $\quad \frac{\cancel{n}\cdot\frac{np}{n-p}}{\cancel{n}-\frac{\cancel{n}p}{n-p}} = \frac{np}{(n-p)(1-\frac{p}{n-p})} = \frac{np}{n-p-p}$

$$\xhookrightarrow[2p>n]{} \mathcal{C}^{0,\,1-\frac{\cancel{n}}{\cancel{n}p/n-p}} \qquad 1-\frac{\cancel{n}}{\cancel{n}p/n-p} = 1-\frac{n-p}{p} = 1+1-\frac{n}{p}$$

Proof of $W^{1,p}(\mathbb{R}^n) \hookrightarrow L^{\frac{np}{n-p}}(\mathbb{R}^n)$; $p<n$

$\boxed{n=2}$

$$\forall g \in \mathcal{S}: \iint |g(x_0,y_0)|^2\, dx_0 dy_0 \le \iint \Big|\int_{-\infty}^{x_0} \partial_x g(x,y_0)\,dx\Big| \Big|\int_{-\infty}^{y_0} \partial_y g(x_0,y)\,dy\Big|\, dx_0 dy_0$$

$$\le \int \Big|\int_{-\infty}^{\infty} \partial_x g(x,y_0)\,dx\Big| dy_0 \cdot \int \Big|\int_{-\infty}^{\infty} \partial_y g(x_0,y)\,dy\Big| dx_0 \le \|\partial_x g\|_{L^1} \|\partial_y g\|_{L^1}$$

$\boxed{\text{similar for all } n}$

$$\Rightarrow \|g\|_{L^{n/n-1}} \le \|\nabla g\|_{L^1}$$

$$g = |h|^\gamma \Rightarrow \|h\|^{\frac{n-1}{n}}_{L^{\gamma\frac{n}{n-1}}} \le \int \gamma |h|^{\gamma-1} |\nabla h| \le \gamma\, \|h\|^{\gamma-1}_{\underset{\text{``}}{L^{p'(\gamma-1)}}} \|\nabla h\|_{L^p}$$

$$= \frac{np}{n-p} \quad \text{for } \gamma = \frac{(n-1)p}{n-p}$$

$$\Rightarrow \|g\|_{L^{\frac{np}{n-p}}} \le \|\nabla g\|_{L^p}$$

So $\underset{\forall' x}{f = \lim_{n} g_i} \in W^{1,p}$, $g_i \in \mathcal{S}$ $\Rightarrow (g_i)$ $L^{\frac{np}{n-p}}$ Cauchy $\Rightarrow$ $\underset{\forall' x \text{ for subsequence}}{f = \lim g_i} \in L^{\frac{np}{n-p}}$ ∎

Proof of $H^s(\mathbb{R}^n) \hookrightarrow \mathcal{C}^{0,\lambda}(\mathbb{R}^n)$ for $\lambda < s - \frac{n}{2}$

$$(2\pi)^n |g(x)-g(y)| = \Big| \int \sqrt{1+|\xi|^2}^{\,s}\, \hat{g}(\xi)\, \sqrt{1+|\xi|^2}^{\,-s} e^{ix\cdot\xi} \underbrace{\left(1-e^{i(y-x)\cdot\xi}\right)}_{|i\sin(y-x)\xi\, +1-\cos(y-x)\xi| \,\le\, 2\,|x-y|^\lambda |\xi|^\lambda} d^n\xi \Big|$$

$$\le \|g\|_{H^s} |x-y|^\lambda \underbrace{\left(\mathrm{Vol}B_1^n + \mathrm{Vol}S_1^{n-1} \int_1^\infty r^{2(\lambda-s)} r^{n-1}\, dr\right)}_{<\infty \text{ if } 2(s-\lambda)>n}$$

$|\sin\alpha| \le \min\{|\alpha|,1\} \le |\alpha|^\lambda \quad \forall\, 0<\lambda\le 1$

same for $1-\cos\alpha$ $(\sim\alpha^2$ for α small, $\le 2) \nearrow$ $|1-\cos\alpha| \le 2\cdot(\frac{2}{\pi})^\lambda |\alpha|^\lambda$

$$\Rightarrow \|g\|_{\mathcal{C}^0} \leq C\|g\|_{H^s} \quad \text{if} \quad s > n/2 \qquad \forall g \in \mathcal{S}$$

$$\|g\|_{\mathcal{C}^{0,\lambda}} \leq C\|g\|_{H^s} \quad \text{if} \quad \lambda < n/2 - s$$

$\rightsquigarrow$ use Cauchy sequences & completeness ∎

So for $g \in H^s$ have $g \sim \tilde{g} \in \mathcal{C}^0$ (i.e. $g(x) = \tilde{g}(x)\ \forall' x \in \mathbb{R}^n$)

Proof of compactness $H^s(\mathbb{R}^n) \longrightarrow H^t(\Omega)$; Ω compact, $s > t$

- H^s is reflexive (since L^2 is) and hence (Banach Alaoglu) $\{\|g\|_{H^s} \leq 1\}$ is weakly compact
- Consider $V_k \xrightarrow{H^s} V_\infty \in H^s(\mathbb{R}^n)$, $\|V_k\|_{H^s} \leq 1$ ($\Rightarrow \|V_\infty\|_{H^s} \leq 1$)

Fix cutoff function $\mu \in \mathcal{C}_c^\infty(\mathbb{R}^n)$, $\mu|_\Omega \equiv 1$ and prove $\|\mu \cdot (V_k - V_\infty)\|_{H^t} \xrightarrow[k\to\infty]{} 0$

then $V_k|_\Omega = \mu V_k|_\Omega \xrightarrow{H^t} \mu V_\infty|_\Omega = V_\infty|_\Omega$.

- $\widehat{\mu(V_k - V_\infty)}(\xi) = \int (V_k - V_\infty) \underbrace{\mu e^{-ix\cdot\xi}}_{\in \mathcal{S} \subset (H^s)^*} d^n x \xrightarrow[k\to\infty]{} 0 \quad \forall \xi$

- $|\widehat{\quad}|(\xi_0) \leq 2 \cdot \|\mu e^{-ix\cdot\xi_0}\|_{H^{-s}} \leq C \sqrt{1+|\xi_0|^2}^{\,-s}$

since $\widehat{\mu e^{-ix\xi_0}}(\xi) = \int \mu(x) e^{-ix\xi_0} e^{-ix\xi} d^n\xi = \hat{\mu}(\xi_0 + \xi)$

$$\Rightarrow \|\mu e^{-ix\cdot\xi_0}\|_{H^{-s}}^2 = \int (1+|\xi|^2)^{-s} |\hat{\mu}(\xi_0+\xi)|^2 d^n\xi = \int \underbrace{(1+|\eta-\xi_0|^2)^{-s}}_{\leq 3^s (1+|\xi_0|^2)^{-s} (1+|\eta|^2)^{+s}} |\hat{\mu}(\eta)|^2 d^n\eta$$

$$\leq C (1+|\xi_0|^2)^{-s} \underbrace{\int (1+|\eta|^2)^s |\hat{\mu}|^2}_{<\infty \text{ since } \mu \in \mathcal{S} \Rightarrow \hat{\mu} \in \mathcal{S}}$$

$$\|\mu(V_k - V_\infty)\|_{H^t}^2 = \int_{B_R} \underbrace{(1+|\xi|^2)^t |\widehat{\mu(V_k - V_\infty)}|^2}_{\leq C(1+|\xi|^2)^{t-s} \chi_R \in L^1} d^n\xi + \underbrace{\int_{\mathbb{R}^n \setminus B_R} (1+|\xi|^2)^t |\widehat{\mu(V_k - V_\infty)}|^2 d^n\xi}_{\leq (1+R^2)^{t-s} \underbrace{\|\mu(V_k - V_\infty)\|_{H^s}}_{\text{bounded } \forall k}} \xrightarrow[k\to\infty]{} 0$$

Dominated Convergence: first term $\xrightarrow{k\to\infty} 0 \quad \forall R > 0$; second term $\xrightarrow{R\to\infty} 0$ ∎

Corollary 1: $f\in W^{m,p}(\Omega),\ \varphi\in \mathcal{C}^m(\mathbb{R}) \Rightarrow \varphi\circ f\in W^{m,p}(\Omega)$ if $mp>n$, Ω compact

(e.g. $\varphi(y)=y^2\sin y \rightsquigarrow \varphi\circ f(x)=f(x)^2\sin f(x)$)

- conditions are fairly sharp since

for $$\underset{\substack{\cap\\ L^r}}{\frac{\partial^m\varphi}{\partial y^m}(f)}\cdot\underset{\substack{\cap\\ L^{\frac{np}{n-(m-1)p}}}}{\partial_{i_1}f}\cdot\ldots\cdot\underset{\substack{\cap\\ \text{or } \mathcal{C}^0 \text{ if } (m-1)p>n}}{\partial_{i_m}f}\in L^p \quad \text{need} \quad \frac{1}{r}+m\cdot\frac{n-(m-1)p}{np}\le\frac{1}{p}$$

$$\Leftrightarrow\ (n\le mp,\ \underset{\substack{\Downarrow\\ \varphi\in\mathcal{C}^m,\ f\in L^\infty}}{r=\infty})\quad\text{or}\quad(n<mp+c(r),\ r<\infty)$$

Proof: for $|\alpha|\le m$ $\partial_\alpha(\varphi\circ f)$ is a sum of terms

$$\frac{\partial^\ell\varphi}{\partial y^\ell}(f)\cdot\partial_{\alpha_1}f\cdot\ldots\cdot\partial_{\alpha_\ell}f \quad\text{with}\quad \sum_{j=1}^{\ell}|\alpha_j|\le m$$

$$\in\ \mathcal{C}^0(\Omega)\cdot\prod_{j=1}^{\ell}\left(L^{\frac{np}{n-(m-|\alpha_j|)p}}\ \text{or}\ \mathcal{C}^0\ \text{if}\ (m-|\alpha_j|)p>n\right)\hookrightarrow L^p$$

if $$\sum_{j=1}^{\ell}\min\left\{\frac{n-(m-|\alpha_j|)p}{np},0\right\}\le\frac{1}{p}$$

$$\le\frac{1}{np}\Big(\ell n-\ell mp+\underbrace{\sum|\alpha_j|}_{\le m}p\Big)\le\frac{1}{p}\,\frac{\ell n-(\ell-1)mp}{n}\le\frac{1}{p}\quad\text{if}\quad n\le mp \qquad \blacksquare$$

Corollary 2: $W^{k,p}$ is a Banach algebra for $kp>n$,

i.e. $W^{k,p}\cdot W^{k,p}\to W^{k,p}$, $(f,g)\mapsto f\cdot g$

is well defined and continuous

Proof: Exercise – similar to Cor. 1

Note: For $\Omega = \mathbb{R}^n$ the Sobolev imbedding for $mp > n$ gives $W^{m+k,p}(\mathbb{R}^n) \hookrightarrow \mathcal{C}_0^k(\mathbb{R}^n)$

since $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}_0^k(\mathbb{R}^n)$ and $(\mathcal{C}_0^k(\mathbb{R}^n), \|\cdot\|_{\mathcal{C}^k})$ complete.

Corollary (i) $\mathcal{S}(\mathbb{R}^n) = \bigcap_{k=0}^{\infty} \sqrt{1+|x|^2}^{-k} H^k(\mathbb{R}^n)$

$\cup$ $\subset$ pick m s.t. $m \cdot 2 > n$

$\bigcap_{k=m}^{\infty} \sqrt{1+|x|^2}^{-k} \mathcal{C}_0^{k-m}(\mathbb{R}^n)$

i.e. $u_{v_\alpha} : \varphi \mapsto \int v_\alpha \cdot \varphi$

(ii) $\mathcal{S}'(\mathbb{R}^n) = \left\{ \sqrt{1+|x|^2}^k \sum_{|\alpha| \le N} D_\alpha v_\alpha \;\middle|\; k, N \in \mathbb{N}, \forall |\alpha| \le N \; v_\alpha \in \mathcal{C}_0^0(\mathbb{R}^n) \right\}$

$= \left\{ \sum_{|\alpha|, |\beta| \le M} x^\alpha D_\beta u_{\alpha\beta} \;\middle|\; M \in \mathbb{N}, \; u_{\alpha\beta} \in \mathcal{C}_0^0(\mathbb{R}^n) \right\} = \left\{ \sum D_\beta (x^\alpha w_{\alpha\beta}) \;\middle|\; \dots \right\}$

Proof of (ii): $u \in \mathcal{S}' \Rightarrow u \in \left(\sqrt{1+|x|^2}^k H^k(\mathbb{R}^n) \right)^*$ for some $k \in \mathbb{N}$

$\Rightarrow \sqrt{1+|x|^2}^{-k} u \in H^{-k}(\mathbb{R}^n) = \left\{ \sum_{|\alpha| \le k+m} D_\alpha v_\alpha \;\middle|\; v_\alpha \in H^m(\mathbb{R}^n) \right\}$

pick $m > \frac{n}{2}$: $\mathcal{C}_0^0(\mathbb{R}^n)$

(Proof as before with $v_\alpha \in L^2$)

$\Rightarrow u = \sqrt{1+|x|^2}^k \sum_\alpha D_\alpha v_\alpha = \dots$ rewrite [Lemma 10.6, Melrose] ■

Corollary: $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ is weakly dense.

I.e. $\forall u \in \mathcal{S}' \; \exists (g_j)_{j \in \mathbb{N}} \subset \mathcal{S} : |u_{g_j}(\varphi) - u(\varphi)| \xrightarrow[j \to \infty]{} 0 \quad \forall \varphi \in \mathcal{S}$

Proof: $u = \sqrt{1+|x|^2}^k \sum D_\alpha v_\alpha$; pick $\mathcal{S} \ni v_\alpha^j \xrightarrow[j \to \infty]{H^m} v_\alpha$

$g_j := \sqrt{1+|x|^2}^k \sum D_\alpha v_\alpha^j$

$$|u_{g_j}(\varphi) - u(\varphi)| = \left| \int_{\mathbb{R}^n} g_j \cdot \varphi - \sum_\alpha \int v_\alpha \cdot (-1)^{|\alpha|} D_\alpha \left(\sqrt{1+|x|^2}^k \varphi \right) \right|$$

$$\le \int_{\mathbb{R}^n} \sum_\alpha \left| (v_\alpha^j - v_\alpha) \cdot \underbrace{(-1)^{|\alpha|} D_\alpha \left(\sqrt{1+|x|^2}^k \varphi \right)}_{\in L^2} \right| \le C_\varphi \sum_\alpha \|v_\alpha^j - v_\alpha\|_{L^2}$$

φ fixed $\downarrow j \to \infty$

0 ■