10 – Fredholm Theory

Morse theory: For $(f : X \to \mathbb{R},\ g \text{ metric on } X)$ "Morse-Smale", $x^\pm \in \mathrm{Crit}\, f$

the Morse trajectory space

$$\widetilde{\mathcal{M}}(x^-, x^+) = \{ \gamma : \mathbb{R} \to X \mid \tfrac{d}{ds}\gamma = -\nabla f(\gamma),\ \gamma(s) \xrightarrow[s \to \pm\infty]{} x^\pm \}$$

is a manifold of dimension $\dim(\text{neg. eigenspace of } \nabla^2 f(x^-)) - \dim(\text{neg. eigenspace of } \nabla^2 f(x^+))$.

$\parallel$ $W^{\text{unstable}}_{x^-} \pitchfork W^{\text{stable}}_{x^+}$

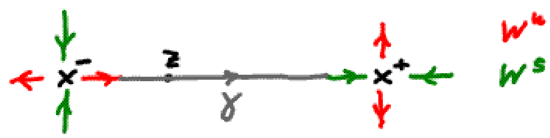
$\parallel$ $\dim \ker \mathrm{Pr}$

$\left(\hat{=} \#\text{eigenvalues of } (\partial_i \partial_j f(\gamma(s)))_{s \in \mathbb{R}} \text{ crossing } 0 \right)$

i.e. $\mathrm{Pr} : T_z W^u_{x^-} \to (T_z W^s_{x^+})^\perp$ onto $\forall z \in W^u_{x^-} \cap W^s_{x^+}$

(see later)

$$\left(\text{index} \quad \dim W^u_{x^-} - \dim (T W^s_{x^+})^\perp = \dim W^u_{x^-} - \dim W^u_{x^+} \right)$$



Goal: Make sense and use of

" The Floer trajectories are the zero set of a Fredholm section

$$\begin{aligned} W^{1,2}(???) &\longrightarrow L^2(???) \\ u &\longmapsto \bar\partial_J u - J X_H(u) \end{aligned}$$

in a Banach bundle,

whose index is the spectral flow of the Hessian." reference: [Robbin-Salamon]

In terms of the linearized operator at a solution u

$$D_u = \frac{\partial}{\partial s} + J_0 \partial_t + S_s + \text{compact perturbation} \quad \text{on } W^{1,2}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

"D_u is Fredholm with $\mathrm{index}(D_u) = \text{spec flow } (J_0 \partial_t + S_s)_{s \in \mathbb{R}}$"

(if surjective then $\parallel$ $\dim \ker D_u$)

$\boxed{\text{Functional Analysis Recap}}$

$\parallel$

"The science of when/how linear algebra works in infinite dimensions"

E, F Banach spaces

$L(E,F) = (\{\text{bounded linear operators } E \to F\}, \text{ operator norm})$

$\cup$

$K(E,F) = \{T \in L(E,F) \text{ compact} : U \subset E \text{ bounded} \Rightarrow \overline{T(U)} \subset F \text{ compact}\}$

Thm: $Id : E \to E$ compact $\Leftrightarrow$ E finite dimensional

Thm: $K(E,F) \subset L(E,F)$ closed subspace

Open Mapping Thm: $T \in L(E,F)$ surjective $\Rightarrow$ T open

(i.e. $U \subset E$ open $\Rightarrow T(U) \subset F$ open)

Corollary $T \in L(E,F)$

a) T bijective $\Leftrightarrow$ T^{-1} bounded

b) T injective, $\operatorname{im} T$ closed $\Leftrightarrow$ $\exists C : \|x\|_E \leq C \|Tx\|_F \quad \forall x \in E$

c) $\underbrace{\begin{pmatrix} \ker T \text{ finite dimensional} \\ \text{and } \operatorname{im} T \text{ closed} \end{pmatrix}}_{T \text{ "semi-Fredholm"}} \Leftrightarrow \begin{pmatrix} \exists K \in K(E,Z), C : \quad \forall x \in E \\ \|x\|_E \leq C(\|Tx\|_F + \|Kx\|_Z) \end{pmatrix}$

Preview: — " —, $\operatorname{codim} \operatorname{im} T = \dim F/_{\operatorname{im} T} < \infty \;:\Leftrightarrow$ T Fredholm

Proof of c) "$\Rightarrow$" uses K = projection to $\ker T$ along $\underbrace{\text{complement}}_{\text{to be defined}}$ $E = \ker T \oplus W$

<u>Proof of Cor. c)</u>

"$\Rightarrow$" $\dim \ker T < \infty \Rightarrow \exists$ complement $E = \ker T \oplus W$

$T|_W : W \to \underbrace{\operatorname{im} T}_{\text{closed}}$ bounded, bijective operator on Banach spaces

$$\Rightarrow \|x\|_E \leq C\left(\|x_W\|_E + \|x_k\|_E\right) \qquad x = x_W + x_k \in W \oplus \ker T$$

$$\leq C\Big(\|T^{-1}\|_{L(\operatorname{im} T; W)} \underbrace{\|Tx_W\|_W}_{= \|Tx\|_F} + \|Pr_{\ker T}\, x\|_{\ker T}\Big)$$

$$\begin{tikzcd} E \arrow[r, "Pr", "\text{bounded}"'] \arrow[rr, bend left, "Pr"] & \ker T \arrow[r, "Id", "\text{compact}"'] & \ker T \end{tikzcd} \text{ : compact}$$

"$\Leftarrow$" • $\dim \ker T < \infty \underset{(x_i) \subset}{\Longleftrightarrow} \{x \in \ker T,\ \|x\| \leq 1\}$ compact

K compact $\Rightarrow \exists$ subsequence (x_i) s.t. $Kx_i \to z \in Z$

$\Rightarrow (x_i)$ Cauchy : $\|x_i - x_j\|_E \leq C\,\|Kx_i - Kx_j\|_Z$

• $\operatorname{im} T$ closed : $E : \ker T \oplus W \to \operatorname{im} T$ bounded, bijective

<u>Claim</u>: $\exists C' : \forall x \in W \quad \|x\|_E \leq C'\,\|Tx\|_F$

$\Rightarrow$ if $Tx_i \to y \in F$; wlog $x_i \in W$, then (x_i) Cauchy,

so $y = \lim Tx_i = T \lim x_i \in \operatorname{im} T$

<u>Proof of Claim by contradiction</u>: $x_i \in W$, $\|x_i\| = 1$, $\|Tx_i\| \to 0$

$\Rightarrow \exists$ subsequence (x_i) s.t. Kx_i converges $\Rightarrow (x_i)$ Cauchy $\quad \|x_i - x_j\| \leq C\left(\|Tx_i - Tx_j\| + \|Kx_i - Kx_j\|\right)$

$\Rightarrow x_i \to x_\infty \in W$, $\|x_\infty\| = 1$, $Tx_\infty = \lim Tx_i = 0$ ↯

Ex: H Hilbert space, $V \subset H$ closed $\Rightarrow H = V \oplus V^\perp$

i.e. $\begin{array}{c} V \times V^\perp \to H \\ (v,w) \mapsto v+w \end{array}$ bijective (always continuous $\Rightarrow$ bounded inverse)

Def$^{\underline{n}}$: E Banach space, $V \subset E$ closed subspace

A complement of V is a closed subspace $W \subset E$ s.t. $E = V \oplus W$

Def$^{\underline{n}}$ (direct sum): $\boxed{E = V \oplus W}$ $:\Leftrightarrow$ $\begin{array}{c} V \times W \to E \\ (v,w) \mapsto v+w \end{array}$ bijective ($\Rightarrow$ bounded inverse)

Facts/Def$^{\underline{ns}}$: E Banach space, $V \subset E$ subspace

- $\dim V < \infty \Rightarrow V$ closed
- $(V, \|\cdot\|_E|_V)$ Banach space $\Leftrightarrow$ V closed
- $\left(E/V,\ \|[w] = w+V\| := \inf_{v \in V} \|w+v\|_E \right)$ Banach space $\Leftrightarrow$ V closed
- $V^\perp := \{ \varphi \in E^* \mid \varphi|_V \equiv 0 \}$ closed
- $E^*/V^\perp \simeq V^*$
- V closed $\Rightarrow V^\perp \simeq (E/V)^*$

Thm: $\dim V < \infty$ ($\Rightarrow V$ closed) or (V closed and $\operatorname{codim} V := \dim(E/V) < \infty$)

$\Rightarrow \exists$ complement W : $E = V \oplus W$

Proof sketch:

- $\dim V < \infty$: pick basis $v_1, \dots, v_n \in V$
  $\rightsquigarrow$ dual basis $\ell_1, \dots, \ell_n \in V^*$
  extend to $L_1, \dots, L_n \in E^*$ by Hahn-Banach
  $W := \bigcap_{i=1}^{n} \ker L_i$

- $\operatorname{codim} V < \infty$: pick basis $\alpha_1, \dots, \alpha_n \in E/V$
  pick representatives $x_1, \dots, x_n \in E$
  $W := \operatorname{span}\{x_1, \dots, x_n\}$

CHECK $E = V \oplus W$

$\boxed{\text{Fredholm theory}}$

Linear Algebra: For $\dim E, \dim F < \infty$, $T \in L(E,F)$

- $\{T \text{ injective}\}, \{T \text{ surjective}\}, \{T \text{ bijective}\} \subset L(E,F)$ open
- $\dim E = \dim \operatorname{im} T + \dim \ker T$ (since $T : E/\ker T \xrightarrow{\sim} \operatorname{im} T$)

$\Rightarrow \dim E - \dim F = \dim \ker T - \operatorname{codim} \operatorname{im} T =: \operatorname{index} T$

Cor: $T \in L(E,E)$ injective $\Leftrightarrow$ surjective

HW: prove / find counterexamples in infinite dimensions (e.g. $E = F = \ell^2$)

For E, F Banach spaces we have

Thm 0 (Fredholm Alternative): $T = Id + K$; $K : E \to E$ compact

$\Rightarrow \dim \ker T = \operatorname{codim} \operatorname{im} T$

Cor: $Id + K$ injective $\Leftrightarrow$ surjective

Def^{n}: $T \in L(E,F)$ **Fredholm** $:\Leftrightarrow \dim \ker T, \operatorname{codim} \operatorname{im} T < \infty$

Rmk: Fredholm $\Rightarrow \operatorname{im} T$ closed by ?

Lemma: If E Banach space, $V \subset E$ subspace, $\dim (E/V) < \infty$, then

V closed $\Leftrightarrow \exists$ Banach space Z, $T \in L(Z,E)$: $V = \operatorname{im} T$

Proof: HW

<u>Ex.:</u> $\Omega \subset \mathbb{R}^n$ compact domain with smooth boundary

$T = \Delta : W_0^{2,2}(\Omega) := \{u \in W^{2,2}(\Omega) \mid u|_{\partial\Omega} = 0\} \longrightarrow L^2(\Omega)$

- $\|u\|_{W^{2,2}} \le C(\|\Delta u\|_{L^2} + \|u\|_{L^2})$ [elliptic regularity with boundary]

  $W^{2,2}(\Omega) \hookrightarrow L^2(\Omega)$ compact so T is semi-Fredholm

- T injective $\left(Tu = 0 \Rightarrow \int_\Omega |\nabla u|^2 = \int_\Omega \Delta u \cdot u = 0 \Rightarrow u = \text{const}\big|_{\partial\Omega} = 0 \Rightarrow u = 0\right)$

  $\Rightarrow \|u\|_{W^{2,2}} \le C' \|\Delta u\|_{L^2}$

- T surjective $\left(\begin{array}{l} \text{variational principle: } u \overset{W_0^{2,2} \to \mathbb{R}}{\longmapsto} \int \frac{1}{2}|\nabla u|^2 - uf \quad \text{bounded below} \\ \rightsquigarrow \text{weak limit of minimizing sequence solves } \Delta u = f \end{array}\right)$

$\Rightarrow T$ Fredholm, index 0

What about $Tu = \Delta u + h \cdot u$; $h \in L^\infty(\Omega)$ $\int \frac{1}{2}(|\nabla u|^2 + hu^2) - fu$

- $h \ge 0 \rightsquigarrow$ variational principle still works $\to T$ bijective
- general h is a "compact perturbation" of Δ, i.e. $T - \Delta : \underset{u \,\mapsto\, h \cdot u}{W_0^{2,2} \to L^2}$ compact

  $\|u\|_{W^{2,2}} \le C(\|\Delta u\|_{L^2} + \|u\|_{L^2}) \le C\big(\|Tu\|_{L^2} + \underbrace{\|h \cdot u\|_{L^2} + \|u\|_{L^2}}_{\text{compact}}\big)$

  $\Rightarrow$ ker T finite dim, im T closed (T semi-Fredholm)

* $T_s = \Delta + s \cdot h$ semi-Fredholm $\forall s \in [0,1]$; T_0 Fredholm, index 0

Kato stability (Thm 2 below) $\Rightarrow T$ Fredholm, index 0 $\Rightarrow T$ surjective iff injective

$\boxed{\text{Main Results}}$

Thm 1: $\{T \text{ Fredholm}\} \subset L(E,F)$ open

$$\underset{\substack{\| \\ \dim\ker - \operatorname{codim} \operatorname{im}}}{\text{index}} : \{T \text{ Fredholm}\} \to \mathbb{Z} \quad \text{continuous}$$

Thm 0 follows from

Thm 2 (Kato stability) $[0,1] \to L(E,F)$, $s \mapsto T_s$ continuous

$$\left.\begin{array}{l} T_0 \text{ Fredholm} \\ T_s \ \underbrace{\text{semi-Fredholm}}_{\Updownarrow} \ \forall s \end{array}\right\} \Rightarrow \begin{array}{l} T_s \text{ Fredholm } \forall s, \\ \text{index } T_s = \text{index } T_0 \end{array}$$

$$\begin{array}{l} \dim\ker T_s < \infty \\ \operatorname{im} T_s \text{ closed} \end{array} \quad \underset{\substack{\text{Open Mapping} \\ \text{Corollary c)}}}{\Longleftrightarrow} \quad \|x\| \le C_s(\|T_s x\| + \|\underset{\text{compact}}{K_s} x\|)$$

Corollary: T Fredholm, K compact $\Rightarrow$ $T+K$ Fredholm, $\operatorname{ind}(T+K) = \operatorname{ind} T$

Proof: $T_s = T + sK$

Proof of Thm 0: $T_s := Id + s \cdot K$

$\ker T_0 = \{0\}$, $\operatorname{im} T_0 = E$ $\Rightarrow$ T_0 Fredholm, index 0

$$\|x\| \le \|x + sKx\| + |s|\,\|Kx\| \le \|T_s x\| + \|Kx\|$$

$$\Rightarrow T_s \text{ semi-Fredholm}$$

Thm 2 $\Rightarrow$ $T_1 = Id + K$ Fredholm, index 0 ∎

<u>Thm</u>: $T \in L(E,F)$

Fredholm	$\Leftrightarrow$	invertible up to compact operators	$\Leftrightarrow$	invertible up to finite dimensions
		$\begin{pmatrix} \exists S \in L(F,E): \\ ST - Id_E \in K(E,E) \\ TS - Id_F \in K(F,F) \end{pmatrix}$		$\begin{pmatrix} \exists\text{splittings } E = K \oplus V,\ F = C \oplus R,\ \dim K, \dim C < \infty \\ \text{such that } pr_R \circ T\|_V \in L(V,R) \text{ invertible} \end{pmatrix}$
(i)		(ii)		(iii)

<u>(i) ⇒ (iii)</u> $K = \ker T$, $R = \operatorname{im} T$ finite dim/codim $\Rightarrow \exists$ complements V, C

$pr_R \circ T|_V : V \hookrightarrow E \xrightarrow{T} F \xrightarrow{pr} R = \operatorname{im} T$ bounded, bijective $\overset{\text{open mapping}}{\Longrightarrow}$ continuous inverse

<u>(ii) ⇒ (i)</u> [assuming Fredholm Alternative] $\ker T \subset \ker \underset{\text{Id + compact}}{\underset{\shortparallel}{ST}}$ finite dim. $\operatorname{im} T \supset \operatorname{im} \underset{\text{Id + compact}}{\underset{\shortparallel}{TS}}$ Finite codim.

<u>(iii) ⇒ (ii)</u> $E \supset V \xrightarrow{pr_R \circ T|_V} R \subset F$, with $S: F \to E$ the reverse map. $S := incl_{V \subset E} \circ (pr_R \circ T|_V)^{-1} \circ pr_R \in L(F,E)$

$$ST = \underbrace{incl \circ (pr_R \circ T|_V)^{-1} \circ pr_R \circ T|_V \circ pr_V}_{pr_V : E \to V \subset E} + S \circ T|_K \circ pr_K$$

$$= Id_E - pr_K + S T|_K \circ \underbrace{pr_K}_{\text{compact}}$$

$$TS = \underbrace{pr_R \circ T|_V \circ (pr_R \circ T|_V)^{-1} \circ pr_R}_{pr_R} + pr_C \circ TS = Id_F - pr_C + \underbrace{pr_C}_{\text{compact}} \circ TS$$

Proof of Thm 1: $S \in \mathrm{Fred}(E,F) \rightsquigarrow E = \ker S \oplus V \quad , F = \operatorname{im} S \oplus \underset{\text{finite dim}}{C}$

$$\begin{aligned} S \oplus Id_C : V \times C &\longrightarrow F \\ (v,c) &\longmapsto Sv + c \end{aligned} \quad \in \{\text{bijective operators } L(V \times C, F)\}$$
open

$\exists \delta > 0 : \|S - T\|_{L(E,F)} \leq \delta \Rightarrow T \oplus Id_C$ bijective

- $T(V) + C = F \Rightarrow \operatorname{codim} \operatorname{im} T \leq \operatorname{codim} \operatorname{im} S < \infty$
- $T|_V$ injective $\Rightarrow \dim \ker T \leq \dim \ker S < \infty$

$\Rightarrow T$ Fredholm

- $\ker T \cap V = \{0\}$, $\operatorname{codim}(\ker T + V) < \infty$

  $\Rightarrow E = \ker T \oplus V \oplus Z$, $\dim \ker T + \dim Z = \dim \ker S$

- $T(V) \cap C = \{0\} \Rightarrow F = T(V) \oplus C \supset \operatorname{im} T = T(V \oplus Z) = T(V) \oplus T(Z)$

  $\Rightarrow \operatorname{codim} \operatorname{im} T = \dim C - \dim T(Z) = \operatorname{codim} \operatorname{im} S - \dim Z$

$\operatorname{ind} T = \operatorname{ind} S \quad \blacksquare$

Remark: Sometimes semi-Fredholm is more generally defined as

$$\operatorname{im} T \text{ closed} \quad \text{and} \quad \left(\dim \ker T < \infty \ \text{ or } \ \operatorname{codim} \operatorname{im} T < \infty \right)$$

with

$$\operatorname{index}(T) \in \underset{\text{Fredholm}}{\mathbb{Z}} \cup \underset{\ker T \text{ infinite dim}}{\{+\infty\}} \cup \underset{\operatorname{im} T \text{ infinite dim}}{\{-\infty\}}$$

Then $\{T \text{ semi-Fredholm}\} \subset L(E,F)$ is open, the index is continuous (locally constant), and this implies Thm 2.

Direct proof of Thm 2 by continuous induction

$$\mathcal{I} := \{ s \in [0,1] \mid T_s \text{ Fredholm}, \operatorname{ind} T_s = \operatorname{ind} T_0 \}$$

$$\left.\begin{array}{l} \bullet \text{ Thm 1} \Rightarrow \text{open} \\ \bullet \ \underline{\text{Claim: } \mathcal{I} \text{ closed}} \end{array}\right\} \ \mathcal{I} \subset [0,1] \text{ connected component} \overset{0 \in \mathcal{I}}{\Longrightarrow} \mathcal{I} = [0,1]$$

Proof: $s_i \in \mathcal{I}$, $s_i \to s_\infty \in [0,1]$

Need to show $\operatorname{codim} \operatorname{im} T_{s_\infty} < \infty$, then $\operatorname{ind} T_{s_\infty} \overset{\text{Thm 1 large } i}{\underset{T_{s_\infty}}{=}} \operatorname{ind} T_{s_i} = \operatorname{ind} T_0$.

- T_{s_∞} semi-Fredholm $\Rightarrow$ $E = \ker T_{s_\infty} \oplus V$, $V \xrightarrow{\sim} \operatorname{im} T_{s_\infty}$

$$\Rightarrow \forall x \in V \quad \|x\|_E \le C \|T_{s_\infty} x\|_F \qquad (\circledast)$$

Claim': $\operatorname{codim} \operatorname{im} T_{s_\infty} = \dim\left(F / T_{s_\infty}(V) \right) \le \dim\left(F / T_{s_i}(V) \right)$ for i suff. large

This proves the Claim since

$$\dim\left(F / T_{s_i}(V) \right) \le \overset{<\infty}{\dim\left(F / \operatorname{im} T_{s_i} \right)} + \dim T_{s_i}\big(\overbrace{\ker T_{s_\infty}}^{\dim < \infty} \big) \qquad < \infty \ \forall i$$

Proof':

Suppose $W \subset F$, $W \cap \operatorname{im} T_{s_\infty} = \{0\}$ $\quad \exists i_n \to \infty : \dim W > \dim(F/T_{s_{i_n}}(V))$
$\dim W < \infty \Rightarrow W$ closed

$T_{s_\infty} \oplus Id_W : \begin{array}{l} V \times W \longrightarrow F \\ (v, w) \longmapsto T_{s_\infty} v + w \end{array}$ is injective

but $T_{s_{i_n}} \oplus Id_W$ cannot be injective $\quad$ ↯ when $\|T_{s_i} - T_{s_\infty}\| \leq \frac{1}{2} C'^{-1}$

Claim'': $\forall v, w: \quad \|v\|_E + \|w\|_F \leq C' \|T_{s_\infty} v + w\|_F$

Suppose not, then find $\|v_k\|_E + \|w_k\|_F = 1$, $\|T_{s_\infty} v_k + w_k\|_F \to 0$

$w_k \longrightarrow w_\infty$ for subsequence since $\dim W < \infty$

$\Rightarrow (T_{s_\infty} v_k) \subset F$ Cauchy $\overset{\circledast}{\Longrightarrow}$ $(v_k) \subset V \subset E$ Cauchy

$\Rightarrow v_k \to v_\infty \in V$, $\quad T_{s_\infty} v_\infty + w_\infty = 0$, $\|v_\infty\|_E + \|w_\infty\|_F = 1$ ↯ injectivity at s_∞

∎