

# Math 140

## Lecture 3

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with modifications by Todor Milev

University of Massachusetts Boston

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## 1 Trigonometry

- Angles
- The Trigonometric Functions
- Trigonometric Identities
- Trigonometric Identities and Complex Numbers
- Graphs of the Trigonometric Functions

# Angles

Angles can be measured in degrees or radians (abbreviated as rad). The angle of a complete rotation contains  $360^\circ$ , which is the same as  $2\pi$  rad. Therefore

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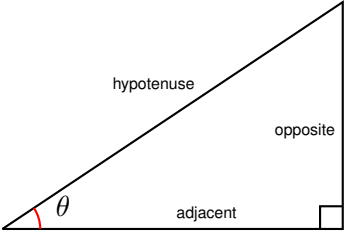
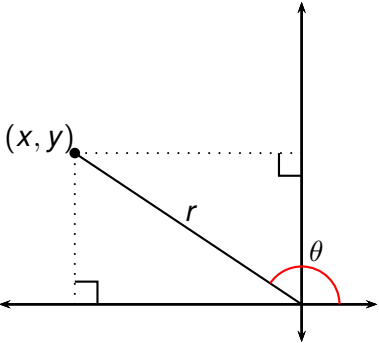
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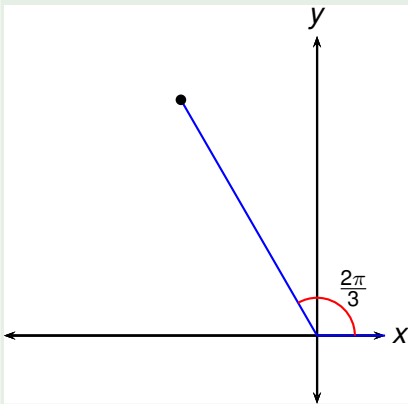
The following table shows the correspondence between degrees and radians for some common angles.

Deg.	$0^\circ$	$30^\circ$	$45^\circ$	$60^\circ$	$90^\circ$	$120^\circ$	$135^\circ$	$150^\circ$	$180^\circ$	$270^\circ$	$360^\circ$
Rad.	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\pi$	$\frac{3\pi}{2}$	$2\pi$

# The Trigonometric Functions

 <p>hypotenuse</p> <p>opposite</p> <p>adjacent</p> <p><math>\theta</math></p>	 <p><math>(x, y)</math></p> <p><math>r</math></p> <p><math>\theta</math></p>
$\sin \theta = \frac{\text{opp}}{\text{hyp}}$ $\csc \theta = \frac{\text{hyp}}{\text{opp}}$ $\cos \theta = \frac{\text{adj}}{\text{hyp}}$ $\sec \theta = \frac{\text{hyp}}{\text{adj}}$ $\tan \theta = \frac{\text{opp}}{\text{adj}}$ $\cot \theta = \frac{\text{adj}}{\text{opp}}$	$\sin \theta = \frac{y}{r}$ $\csc \theta = \frac{r}{y}$ $\cos \theta = \frac{x}{r}$ $\sec \theta = \frac{r}{x}$ $\tan \theta = \frac{y}{x}$ $\cot \theta = \frac{x}{y}$
Acute angles	Obtuse or negative angles

## Example



Find the exact trigonometric ratios for  $\theta = 2\pi/3 = 120^\circ$ .

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$$\cos \frac{2\pi}{3} =$$

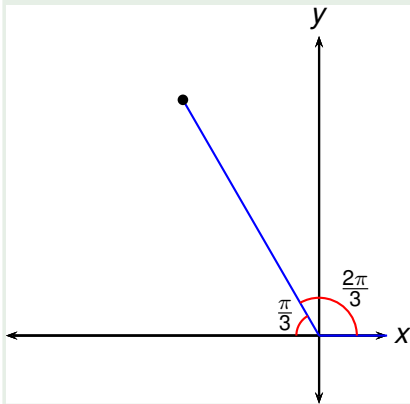
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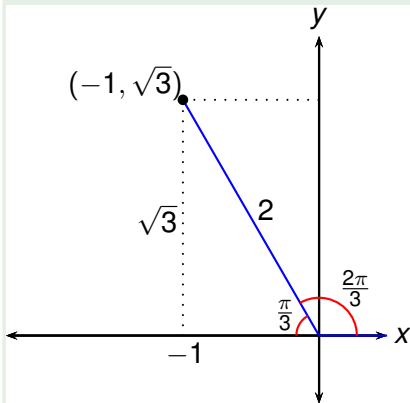
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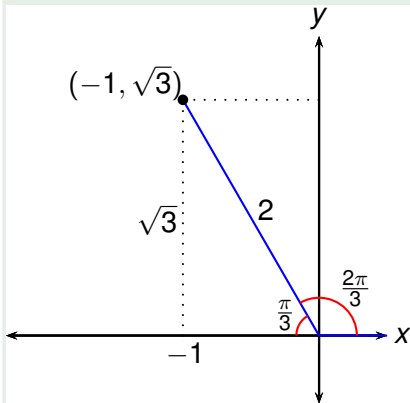
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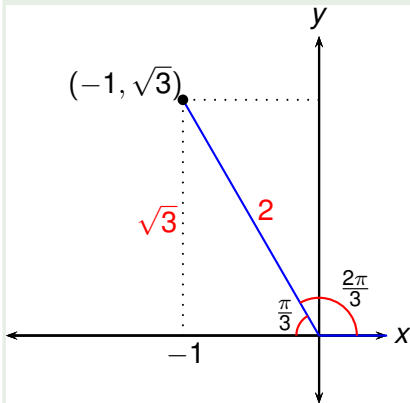
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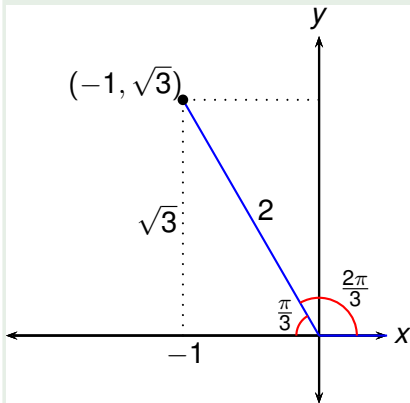
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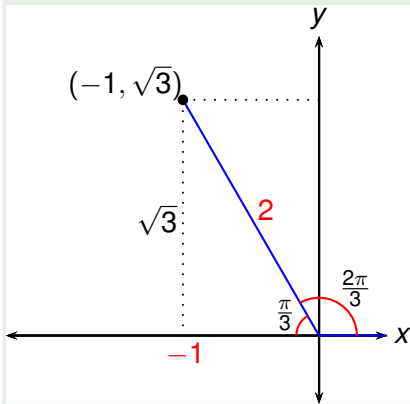
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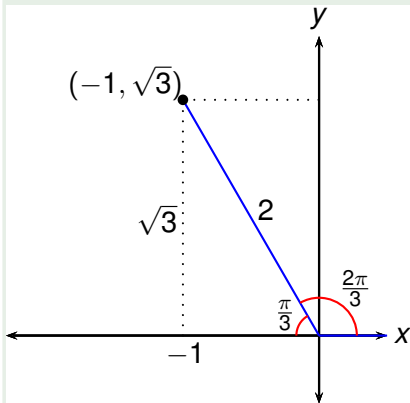
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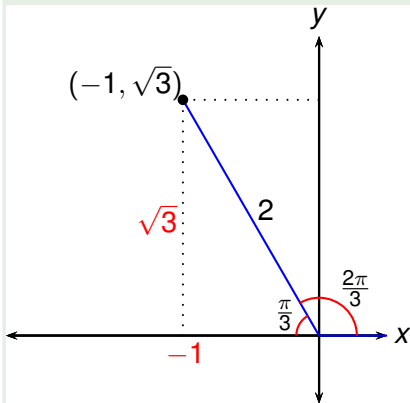
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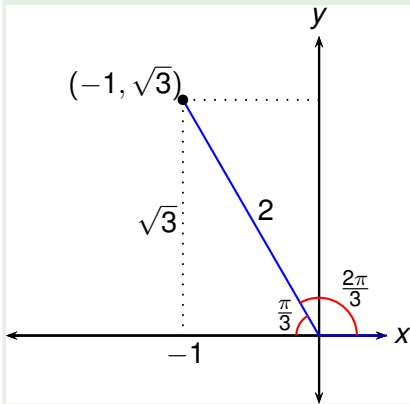
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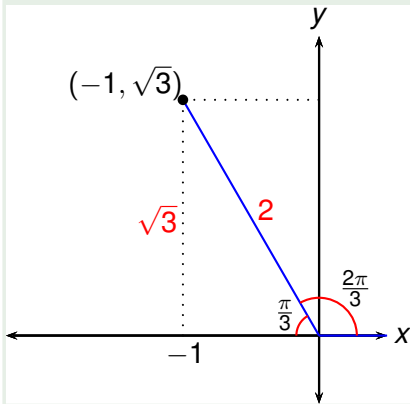
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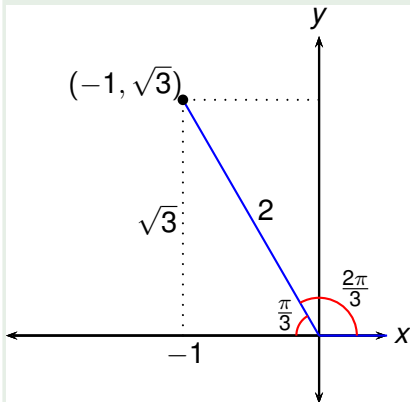
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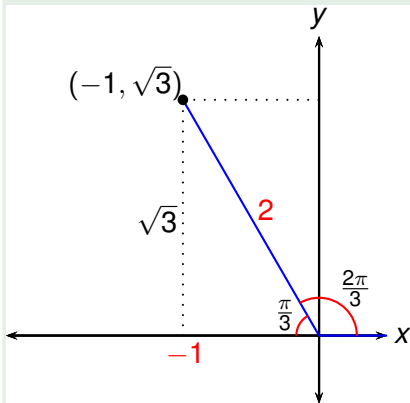
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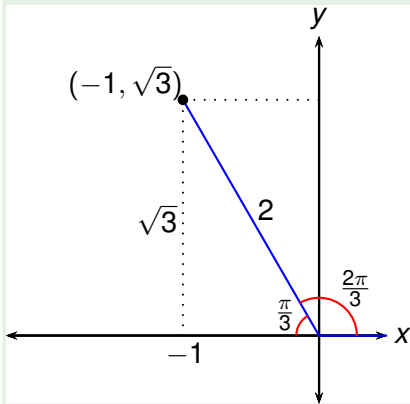
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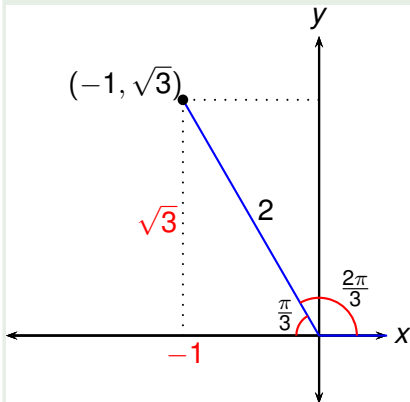
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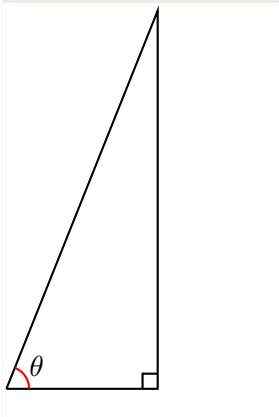
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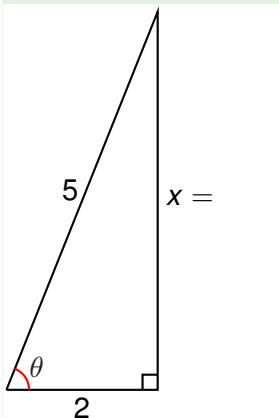
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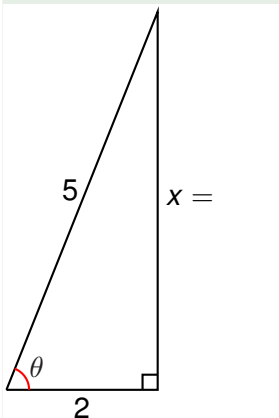
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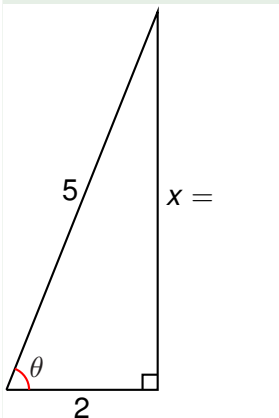
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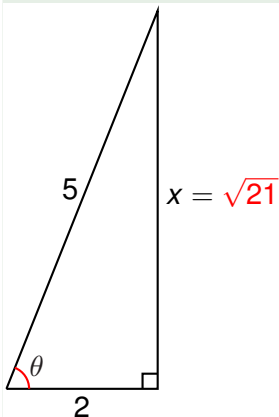
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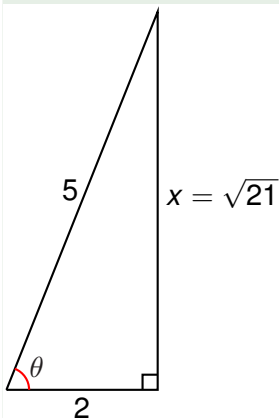
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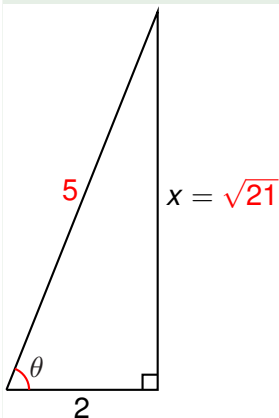
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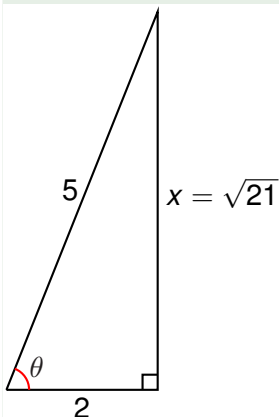
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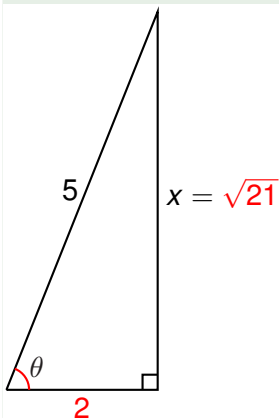
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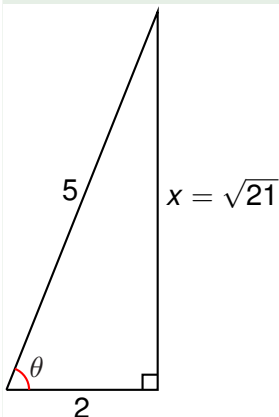
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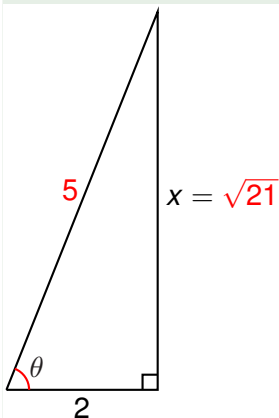
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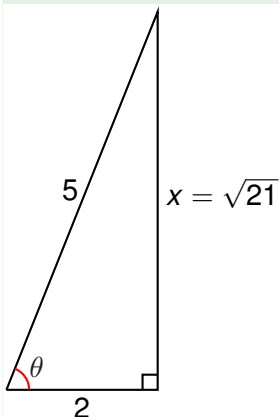
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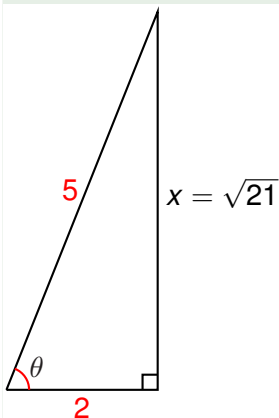
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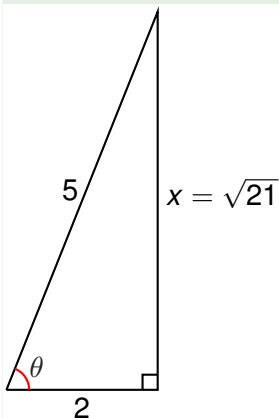
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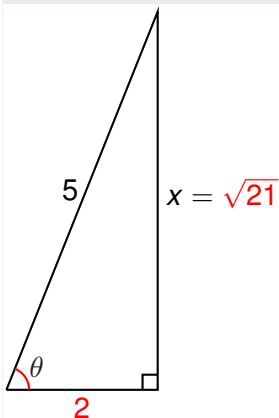
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$$\sin \theta = \frac{\sqrt{21}}{5} \quad \tan \theta = \frac{\sqrt{21}}{2}$$

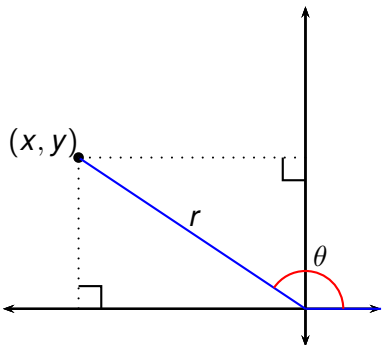
$$\csc \theta = \frac{5}{\sqrt{21}} \quad \sec \theta = \frac{5}{2}$$

$$\cot \theta = \frac{2}{\sqrt{21}}$$

# Trigonometric Identities

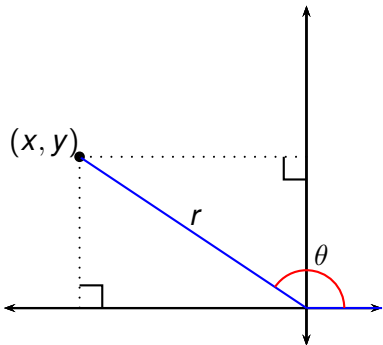
## Definition (Trigonometric Identity)

A trigonometric identity is a relationship among the trigonometric functions that is true for any value of the independent variable.

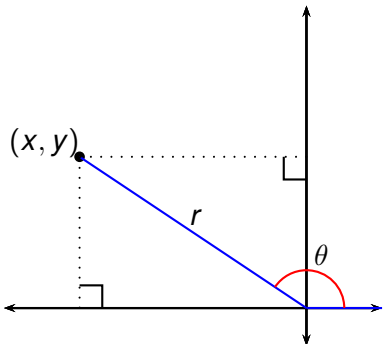


$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

- $\csc \theta = \frac{1}{\sin \theta}$
- $\sec \theta = \frac{1}{\cos \theta}$
- $\cot \theta = \frac{1}{\tan \theta}$
- $\tan \theta = \frac{\sin \theta}{\cos \theta}$
- $\cot \theta = \frac{\cos \theta}{\sin \theta}$

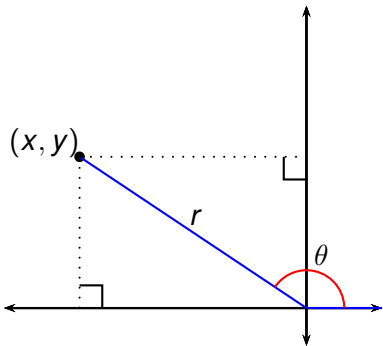


$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



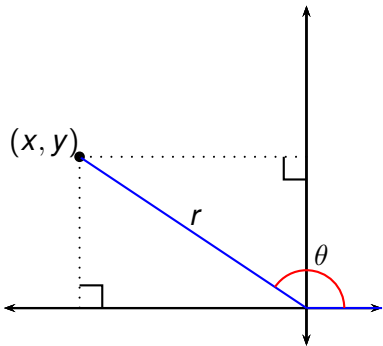
$$\sin^2 \theta + \cos^2 \theta$$

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



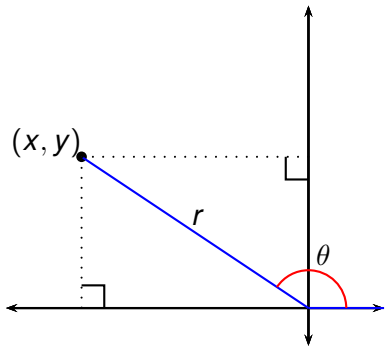
$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

$$= \frac{\sin^2 \theta + \cos^2 \theta}{\frac{y^2}{r^2} + \frac{x^2}{r^2}}$$



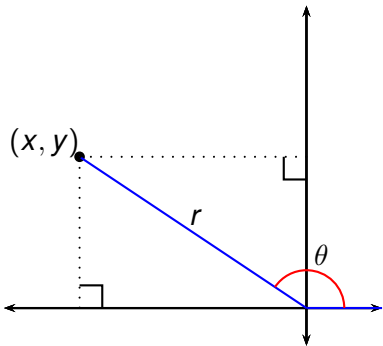
$$\begin{aligned}\sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y}\end{aligned}$$

$$\begin{aligned}& \sin^2 \theta + \cos^2 \theta \\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ &= \frac{y^2 + x^2}{r^2}\end{aligned}$$



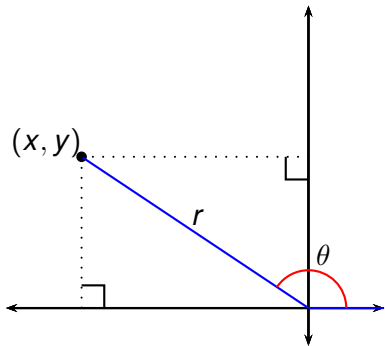
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$$\begin{aligned} & \sin^2 \theta + \cos^2 \theta \\ = & \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ = & \frac{y^2 + x^2}{r^2} \\ = & \frac{r^2}{r^2} \end{aligned}$$



$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

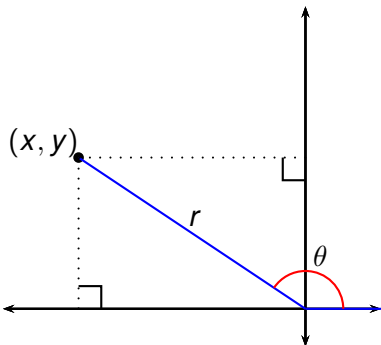
$$\begin{aligned} & \sin^2 \theta + \cos^2 \theta \\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ &= \frac{y^2 + x^2}{r^2} \\ &= \frac{r^2}{r^2} \\ &= 1 \end{aligned}$$



$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

$$\begin{aligned} & \sin^2 \theta + \cos^2 \theta \\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2} \\ &= \frac{y^2 + x^2}{r^2} \\ &= \frac{r^2}{r^2} \\ &= 1 \end{aligned}$$

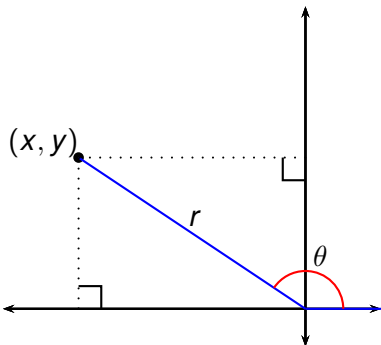
Therefore  $\sin^2 \theta + \cos^2 \theta = 1$ .



$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

**Example** ( $\tan^2 \theta + 1 = \sec^2 \theta$ )

Prove the identity  
 $\tan^2 \theta + 1 = \sec^2 \theta$ .



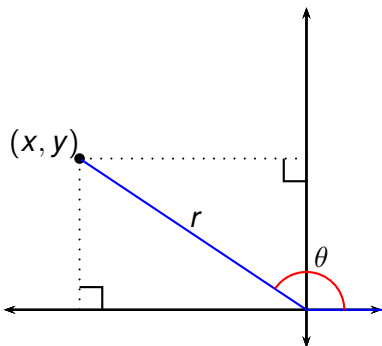
$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

### Example ( $\tan^2 \theta + 1 = \sec^2 \theta$ )

Prove the identity

$$\tan^2 \theta + 1 = \sec^2 \theta.$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

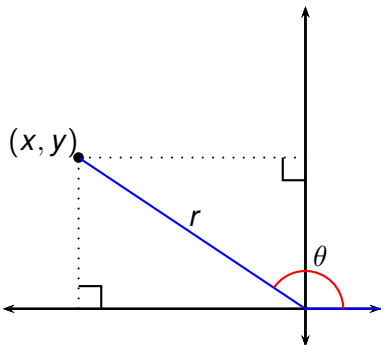


$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

### Example ( $\tan^2 \theta + 1 = \sec^2 \theta$ )

Prove the identity  
 $\tan^2 \theta + 1 = \sec^2 \theta$ .

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} &= \frac{1}{\cos^2 \theta} \end{aligned}$$

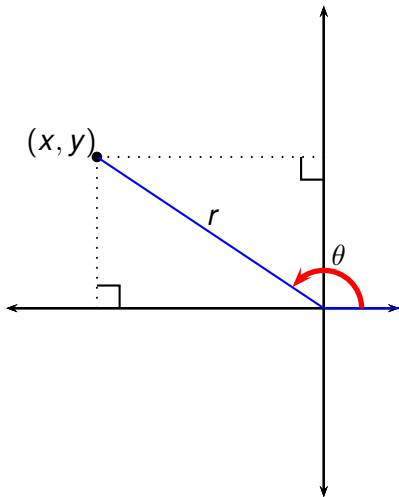


$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

### Example ( $\tan^2 \theta + 1 = \sec^2 \theta$ )

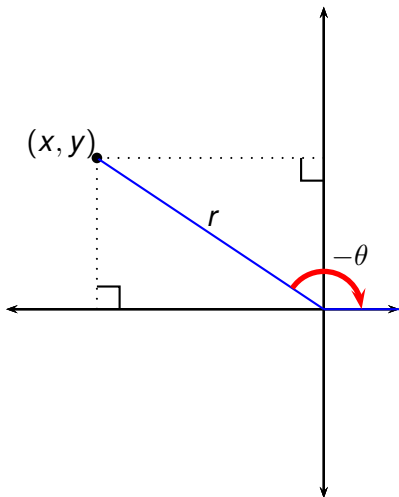
Prove the identity  
 $\tan^2 \theta + 1 = \sec^2 \theta$ .

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= 1 \\ \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} &= \frac{1}{\cos^2 \theta} \\ \tan^2 \theta + 1 &= \sec^2 \theta \end{aligned}$$



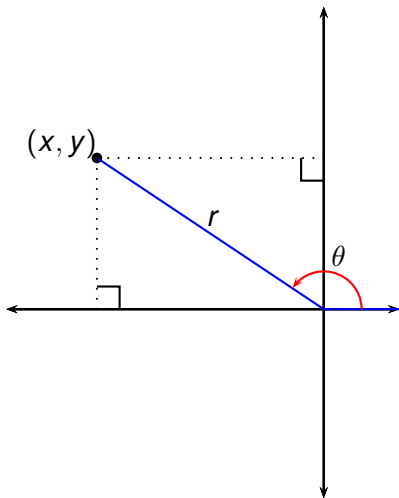
- Positive angles are obtained by rotating counterclockwise.

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



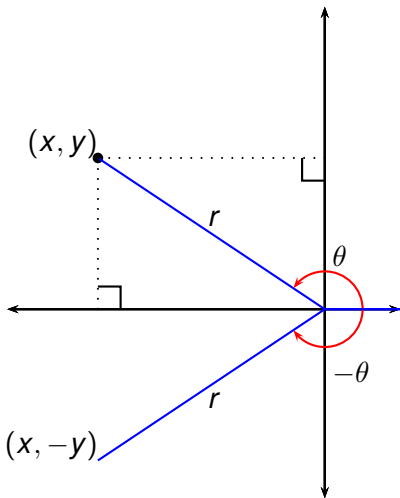
- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.

$$\begin{aligned}\sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y}\end{aligned}$$



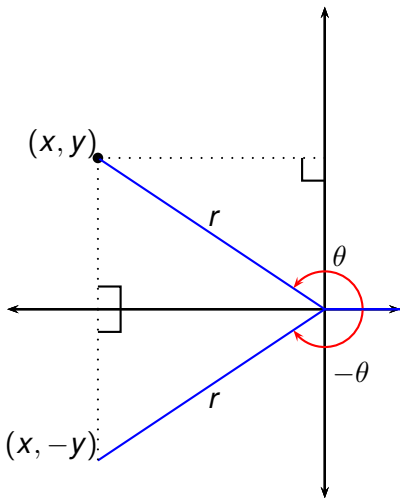
- Positive angles are obtained by rotating counterclockwise.
- Negative angles are obtained by rotating clockwise.
- If  $(x, y)$  is on the terminal arm of the angle  $\theta$ , then  $(x, -y)$  is on the terminal arm of  $-\theta$ .

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



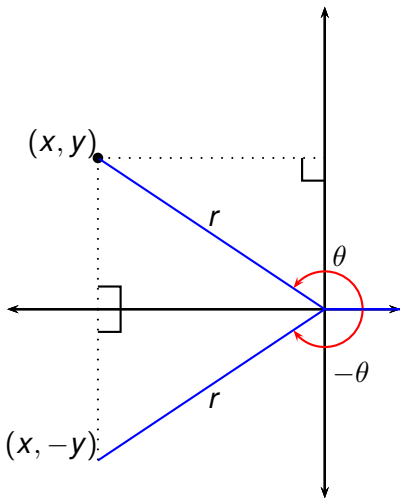
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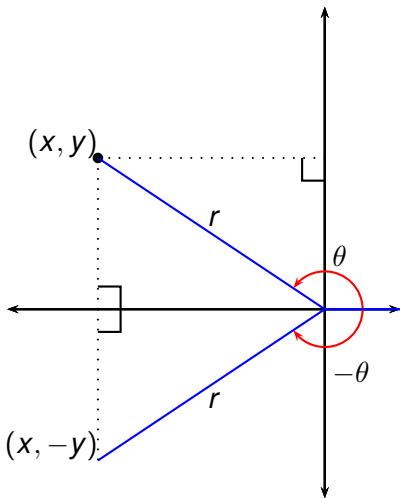
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$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



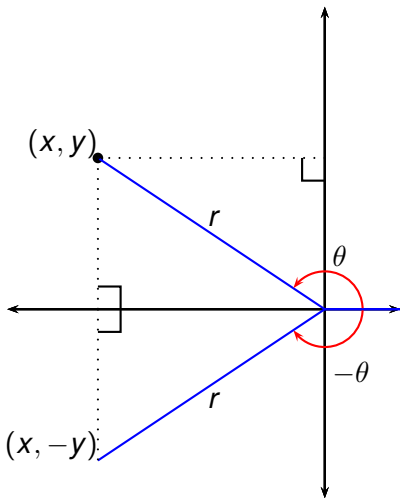
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$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



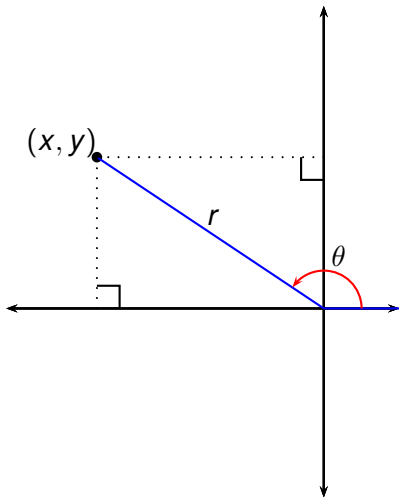
$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

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- $\cos(-\theta) = \frac{x}{r} = \cos \theta$ .
- $\sin$  is an **odd function**.

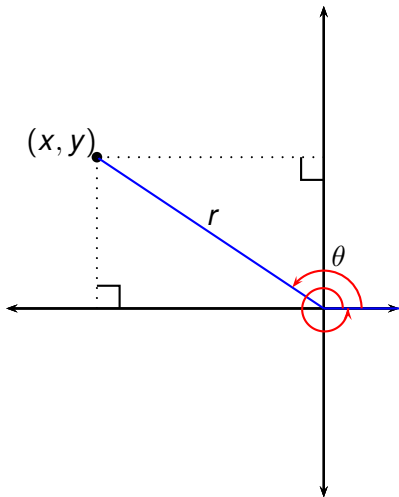


$$\begin{aligned} \sin \theta &= \frac{y}{r} & \csc \theta &= \frac{r}{y} \\ \cos \theta &= \frac{x}{r} & \sec \theta &= \frac{r}{x} \\ \tan \theta &= \frac{y}{x} & \cot \theta &= \frac{x}{y} \end{aligned}$$

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- $\cos(-\theta) = \frac{x}{r} = \cos \theta$ .
- $\sin$  is an odd function.
- $\cos$  is an even function.

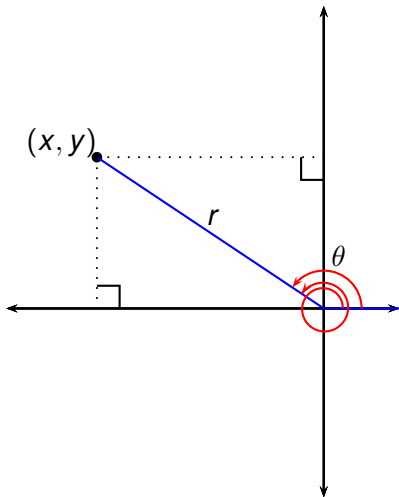


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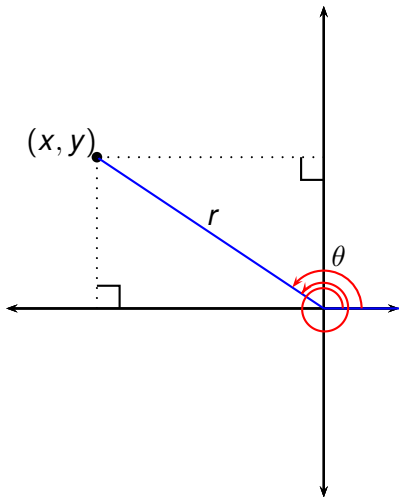
- $2\pi$  represents a full rotation.

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



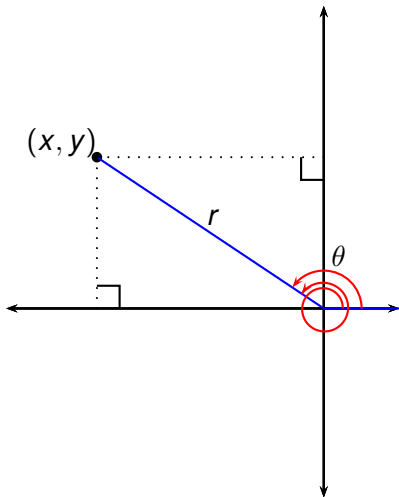
- $2\pi$  represents a full rotation.
- $\theta + 2\pi$  has the same terminal arm as  $\theta$ .

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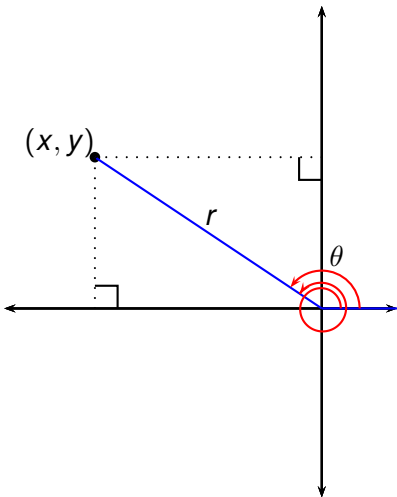
- $2\pi$  represents a full rotation.
- $\theta + 2\pi$  has the same terminal arm as  $\theta$ .
- $\theta + 2\pi$  uses the same point  $(x, y)$  and the same length  $r$ .

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



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- $\theta + 2\pi$  has the same terminal arm as  $\theta$ .
- $\theta + 2\pi$  uses the same point  $(x, y)$  and the same length  $r$ .
- $\sin(\theta + 2\pi) = \sin \theta$ .
- $\cos(\theta + 2\pi) = \cos \theta$ .

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$



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- $\sin(\theta + 2\pi) = \sin \theta$ .
- $\cos(\theta + 2\pi) = \cos \theta$ .
- We say sin and cos are  $2\pi$ -periodic.

$$\begin{array}{ll} \sin \theta = \frac{y}{r} & \csc \theta = \frac{r}{y} \\ \cos \theta = \frac{x}{r} & \sec \theta = \frac{r}{x} \\ \tan \theta = \frac{y}{x} & \cot \theta = \frac{x}{y} \end{array}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

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Substitute  $-y$  for  $y$ , and use the fact that  $\sin(-y) = -\sin y$  and  $\cos(-y) = \cos y$ :

$$\begin{aligned}\sin(x - y) &= \sin x \cos y - \cos x \sin y \\ \cos(x - y) &= \cos x \cos y + \sin x \sin y\end{aligned}$$

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The remaining identities are consequences of the addition formulas:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

To get the double angle formulas, substitute  $x$  for  $y$ :

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

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To get the double angle formulas, substitute  $x$  for  $y$ :

$$\begin{aligned}\sin 2x &= 2 \sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x\end{aligned}$$

Rewrite the second double angle formula in two ways, using  $\cos^2 x = 1 - \sin^2 x$  and  $\sin^2 x = 1 - \cos^2 x$ :

$$\begin{aligned}\cos 2x &= 2 \cos^2 x - 1 \\ \cos 2x &= 1 - 2 \sin^2 x\end{aligned}$$

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$$\begin{aligned}\cos 2x &= 2 \cos^2 x - 1 \\ \cos 2x &= 1 - 2 \sin^2 x\end{aligned}$$

To get the half-angle formulas, solve these equations for  $\cos^2 x$  and  $\sin^2 x$  respectively.

$$\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sin^2 x = \frac{1 - \cos 2x}{2}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

Divide the first equation by the second, and then cancel  $\cos x \cos y$  from the top and bottom:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

The remaining identities are consequences of the addition formulas:

$$\begin{aligned}\sin(x + y) &= \sin x \cos y + \cos x \sin y \\ \cos(x + y) &= \cos x \cos y - \sin x \sin y\end{aligned}$$

Divide the first equation by the second, and then cancel  $\cos x \cos y$  from the top and bottom:

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Do the same for the subtraction formulas:

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

## Example

Find all values of  $x$  in the interval  $[0, 2\pi]$  such that  $\sin x = \sin 2x$ .

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## Example

Find all values of  $x$  in the interval  $[0, 2\pi]$  such that  $\sin x = \sin 2x$ .

$$\sin x = \sin 2x$$

$$\sin x = 2 \sin x \cos x$$

$$0 = 2 \sin x \cos x - \sin x$$

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Find all values of  $x$  in the interval  $[0, 2\pi]$  such that  $\sin x = \sin 2x$ .

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Therefore the equation has five solutions:  $0, \frac{\pi}{3}, \pi, \frac{5\pi}{3},$  and  $2\pi$ .

# Complex numbers definition

## Definition

The set of complex numbers  $\mathbb{C}$  is defined as the set

$$\{a + bi \mid a, b - \text{real numbers}\},$$

where the number  $i$  is a number for which

$$i^2 = -1 \quad .$$

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# Euler's Formula

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# Euler's Formula

## Theorem (Euler's Formula)

$$e^{ix} = \cos x + i \sin x,$$

where  $e \approx 2.71828$  is Euler's/Napier's constant.

## Proof.

Recall  $n! = 1 * 2 * 3 * \dots * (n - 1) * n$ . Borrow from Calc II the f-las:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots + \frac{(-1)^n x^{2n}}{(2n)!} + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

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$e^z$	=	1	+	z	+	$\frac{z^2}{2!}$	+	$\frac{z^3}{3!}$	+	$\frac{z^4}{4!}$	+	$\frac{z^5}{5!}$	+	...

Rearrange.

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Rearrange. **Plug-in  $z = ix$ .**

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Rearrange. Plug-in  $z = ix$ . Use  $i^2 = -1$ . Multiply  $\sin x$  by  $i$ . **Add to get**  
 $e^{ix} = \cos x + i \sin x.$  □

# Trigonometric Identities Revisited

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Compare coefficient in front of  $i$  and **remaining terms** to get the desired equalities.

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$$\begin{aligned}e^{i(2x)} &= \cos 2x + i \sin 2x \\ e^{ix} e^{ix} &= \cos 2x + i \sin 2x\end{aligned}$$

# Trigonometric Identities Revisited

All trigonometric formulas can be easily derived using

- $e^{ix} = \cos x + i \sin x$  (Euler's Formula).
- $e^{ix} e^{iy} = e^{ix+iy} = e^{i(x+y)}$  (exponentiation rule: valid for  $\mathbb{C}$ ).
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## Example

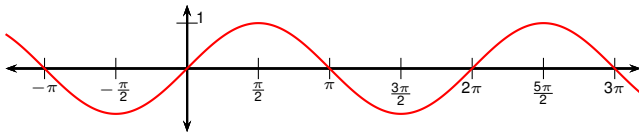
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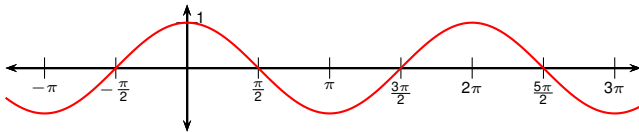
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Compare coefficient in front of  $i$  and **remaining terms** to get the desired equalities.

# Graphs of the Trigonometric Functions

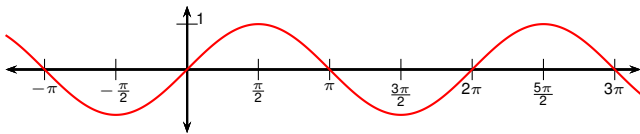


$$y = \sin x$$

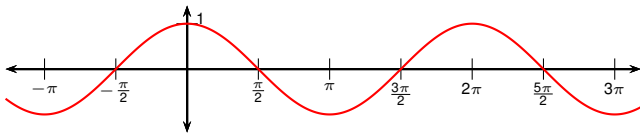


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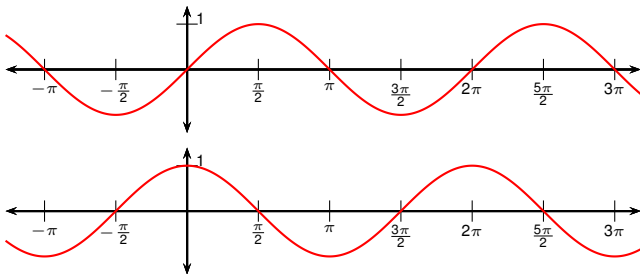
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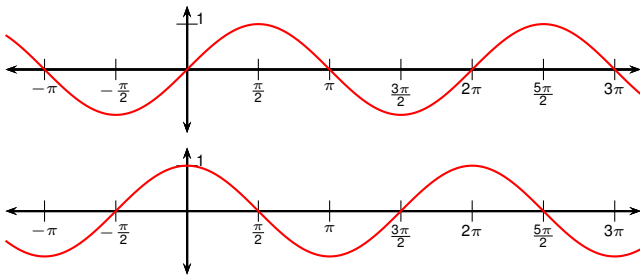


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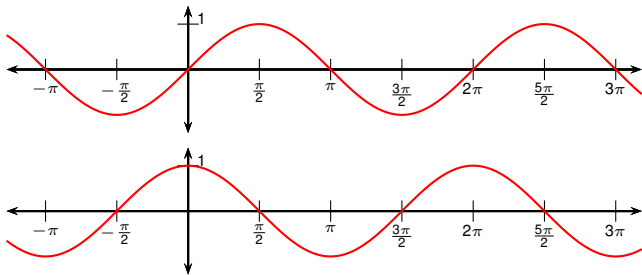


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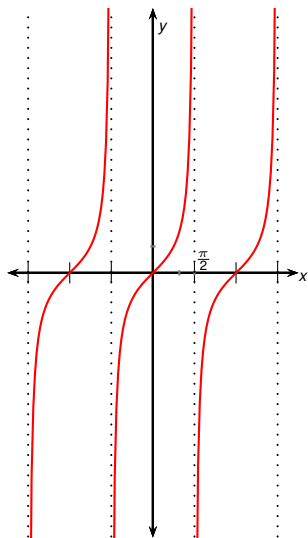
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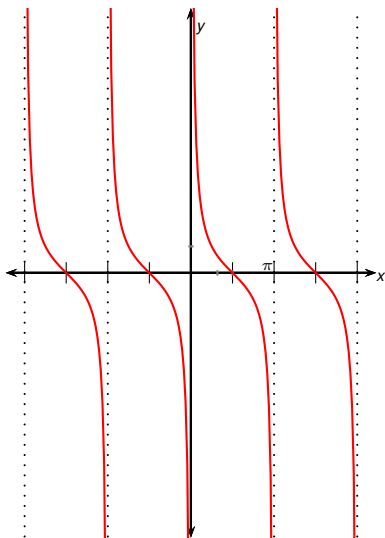
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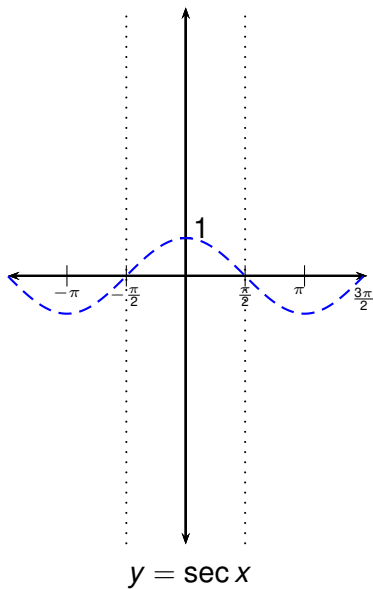
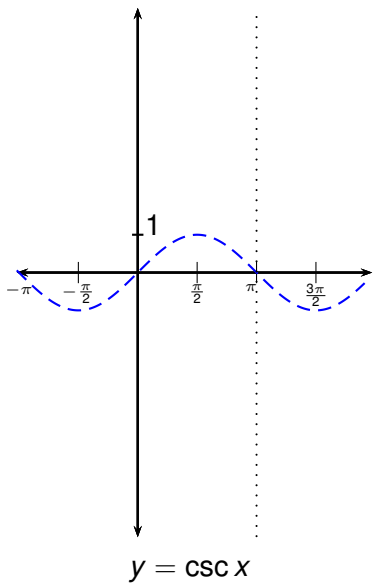
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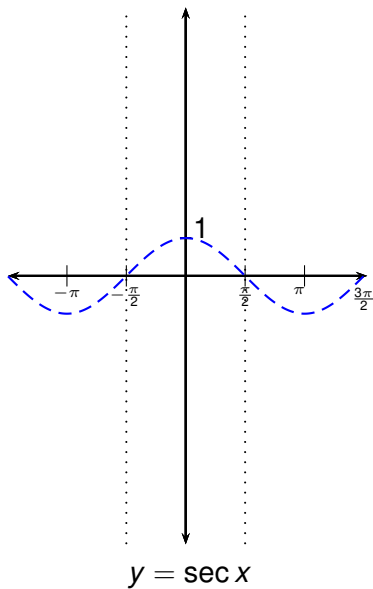
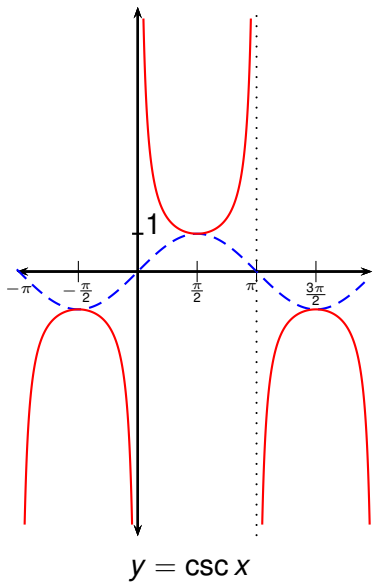


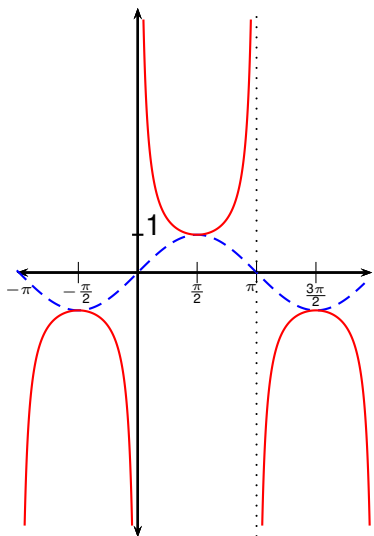
$$y = \tan x$$



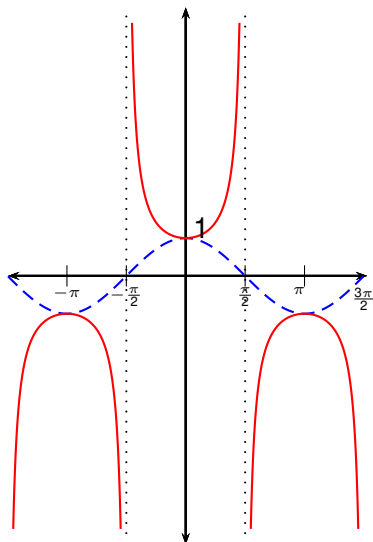
$$y = \cot x$$







$$y = \csc x$$



$$y = \sec x$$