Math 140

Lecture 9

Greg Maloney

with modifications by Todor Milev

University of Massachusetts Boston

September 30-October 4, 2013

Outline

1. Tangents

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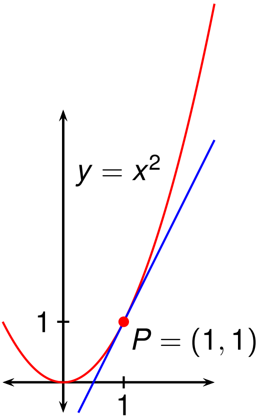

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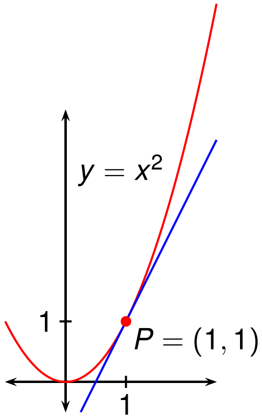
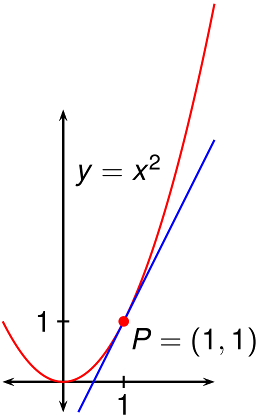
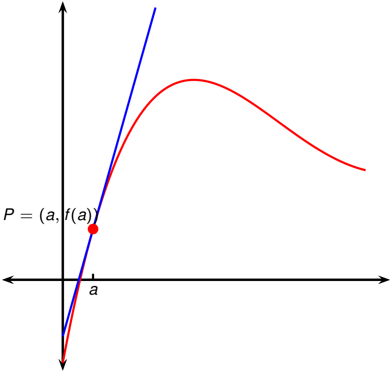


- Recall that in section (2.1) we tried to find the tangent line to the curve $y = x^2$ at the point $P = (1, 1)$.

Tangents



- Recall that in section (2.1) we tried to find the tangent line to the curve $y = x^2$ at the point $P = (1, 1)$.
- This problem motivated us to study limits.



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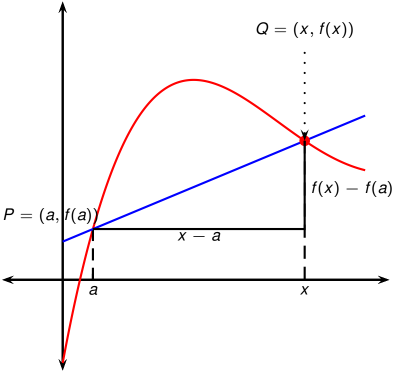

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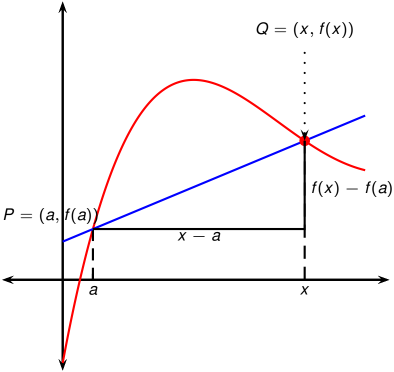

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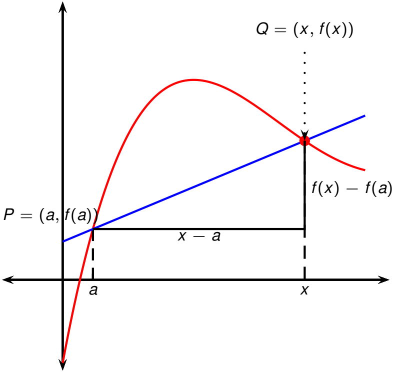

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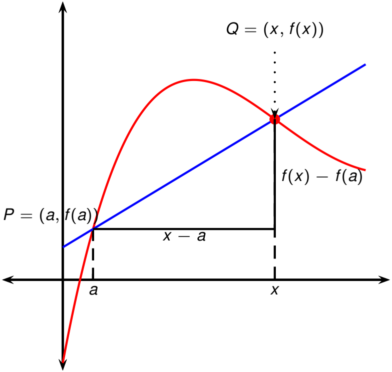

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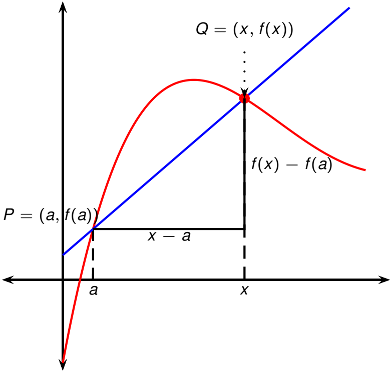

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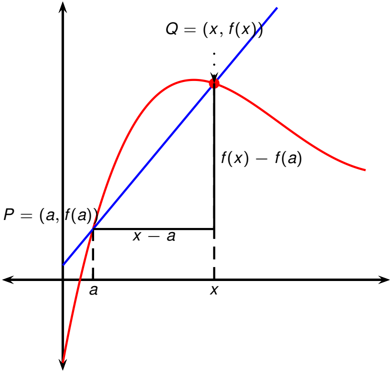

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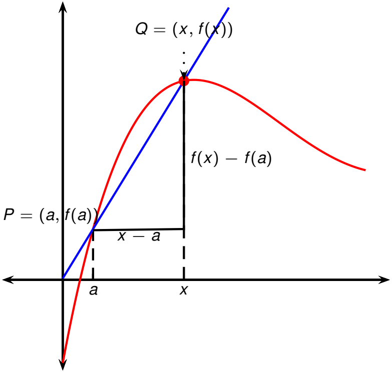

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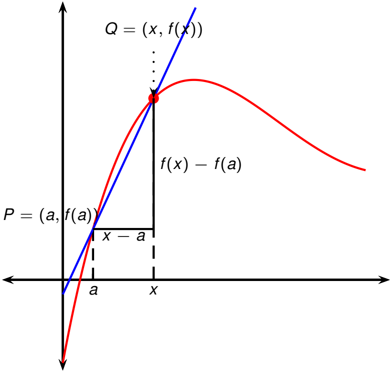

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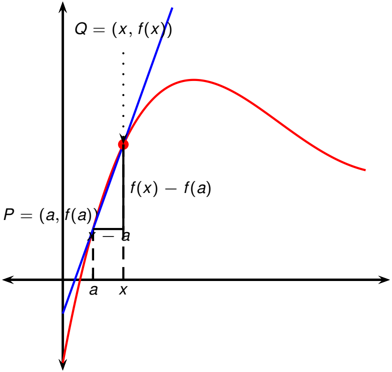

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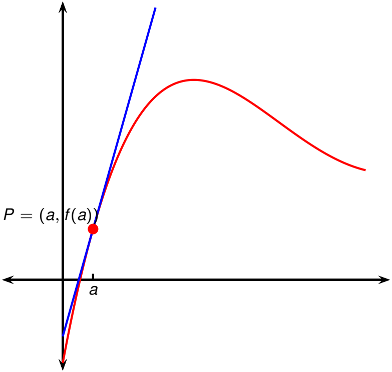

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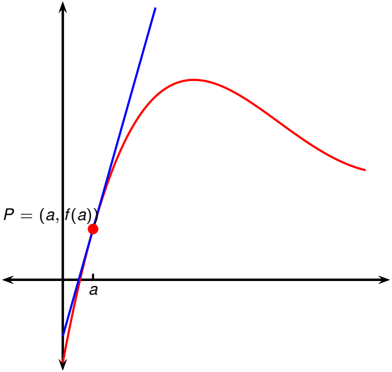
Definition (Non-vertical tangent line)

Let $P = (a, f(a))$. Suppose the limit $m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists. Define the tangent to $y = f(x)$ at P to be the line passing through P with slope m



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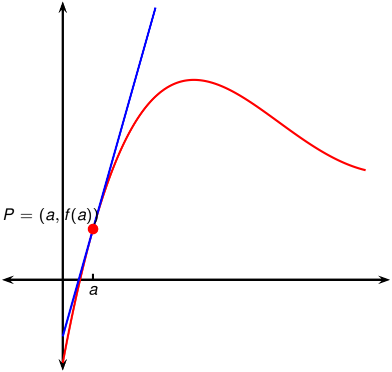

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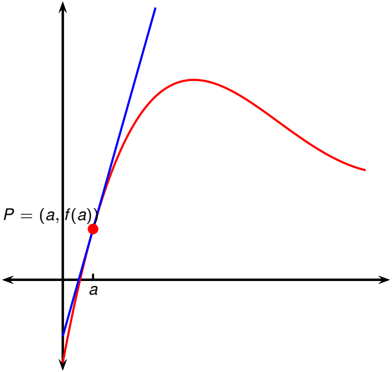

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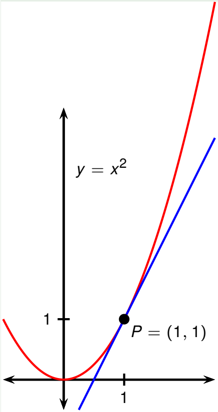
Note. If the limit does not exist, we give no definition of a tangent line.

Example

Find an equation for the tangent line to the parabola $y = x^2$ at the point $P = (1, 1)$.



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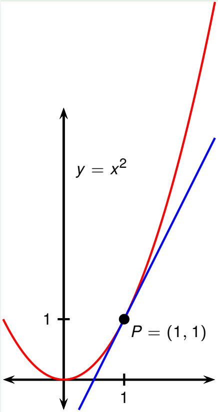

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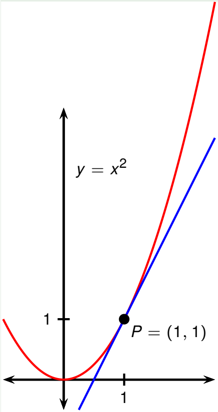

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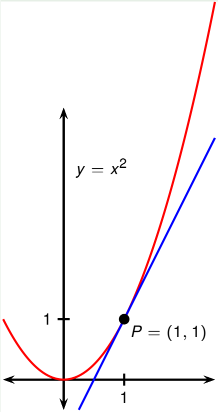

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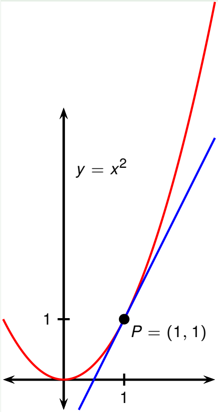

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$$\begin{aligned} m &= \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \\ &= \lim_{x \to 1} \frac{(x-1)(x+1)}{x - 1} \end{aligned}$$

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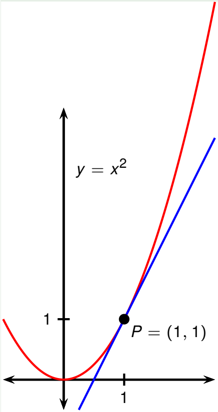

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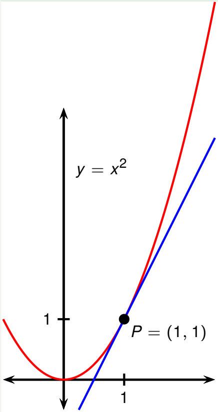

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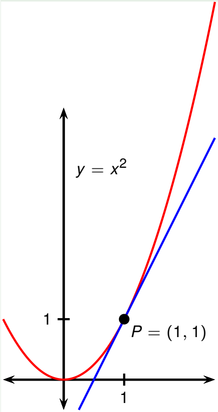

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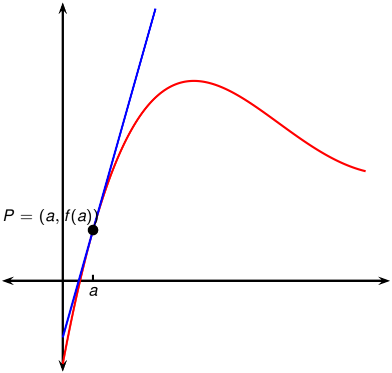
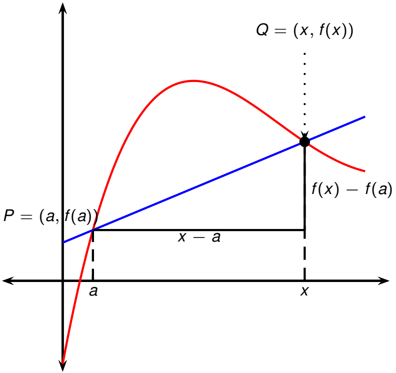
Point-slope form: $y - 1 = 2(x - 1)$, or $y = 2x - 1$.



- There is an equivalent expression for the slope of the tangent.



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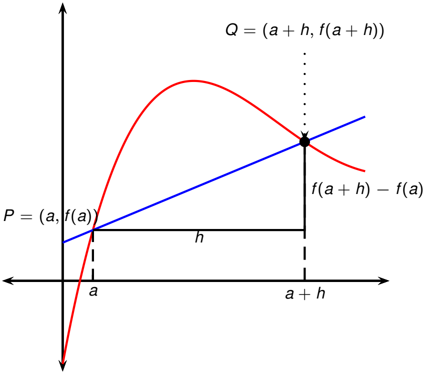

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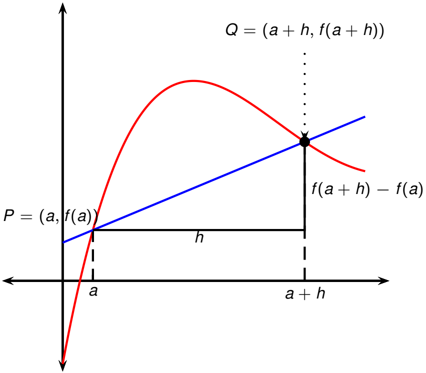

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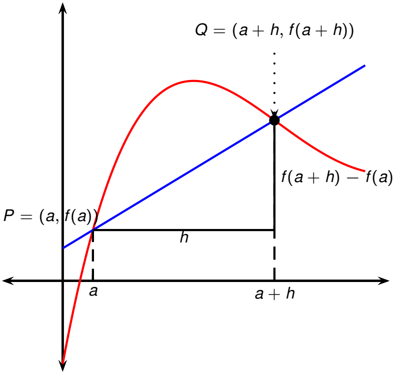

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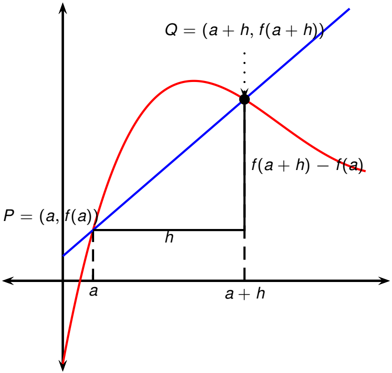

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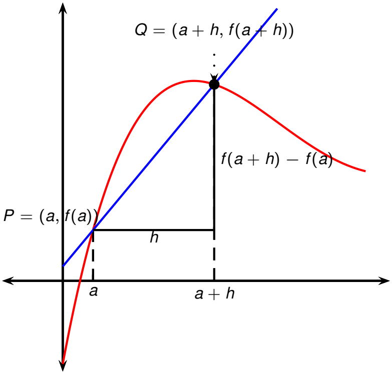

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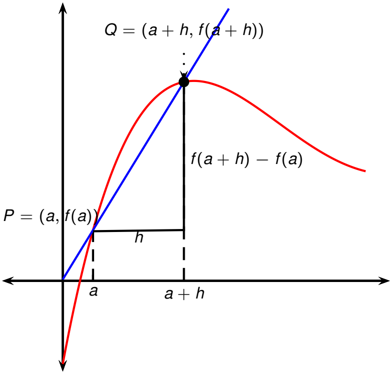

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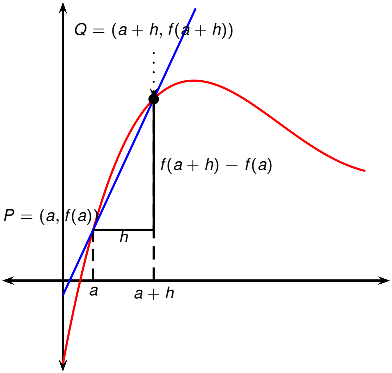

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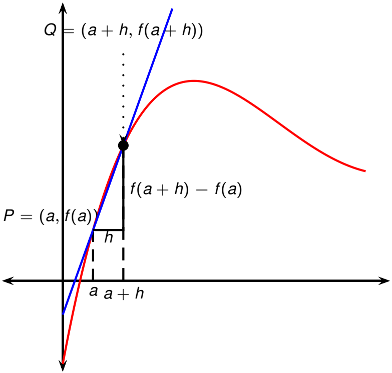

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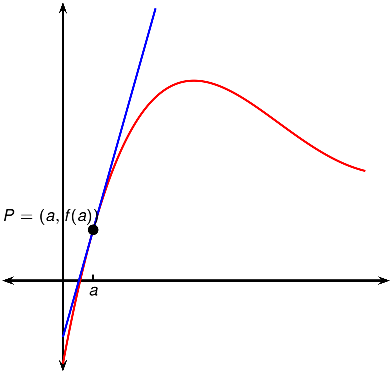

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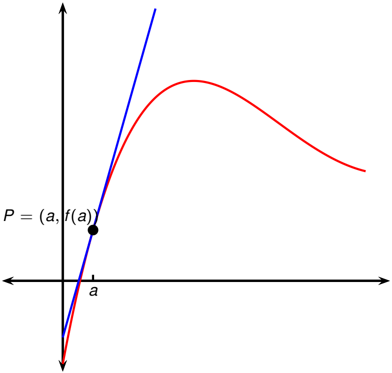

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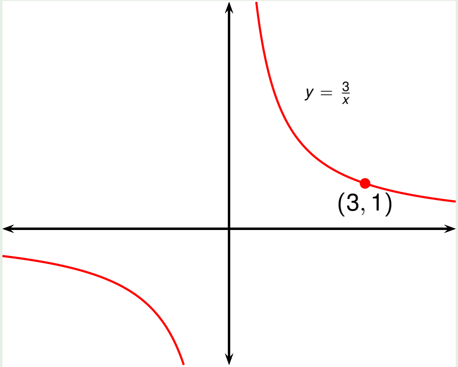
Tangent slope - equivalent expression:

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

Example

Find an equation for the tangent line to the hyperbola $y = 3/x$ at the point $(3, 1)$.



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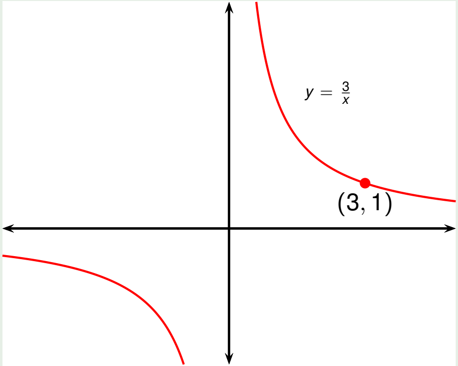

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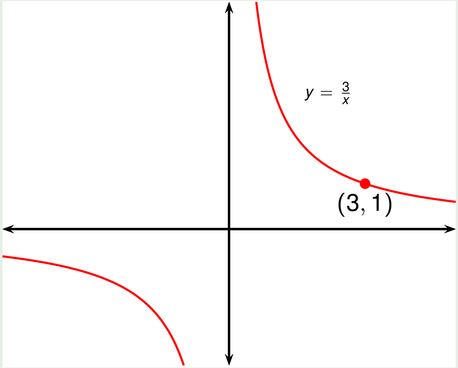

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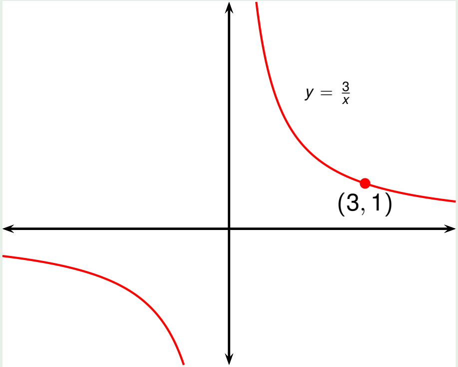

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$$m = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h} = \lim_{h\to 0} \frac{\frac{3}{3+h} - 1}{h}$$

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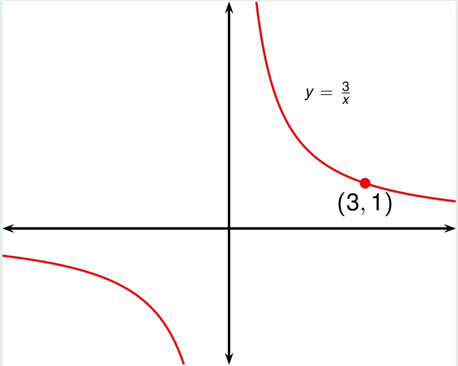

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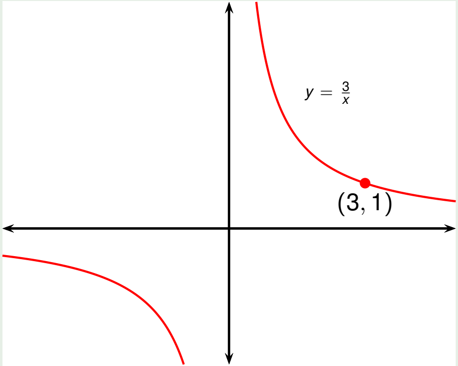

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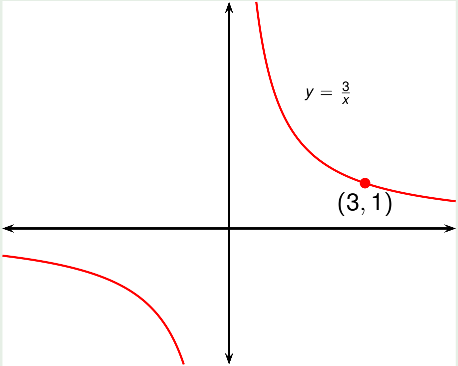

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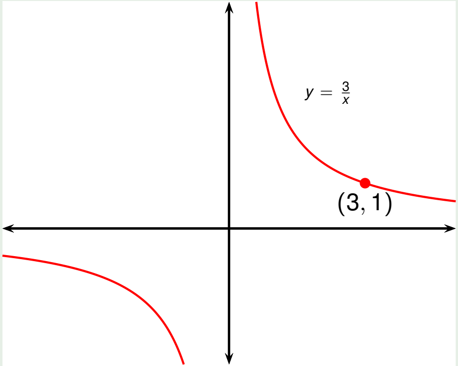

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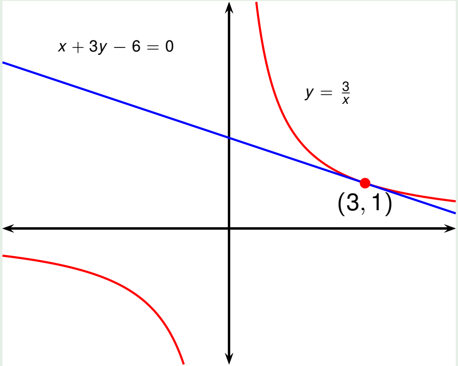

Point-slope form: $y - 1 = -\frac{1}{3}(x - 3)$, or $x + 3y - 6 = 0$.

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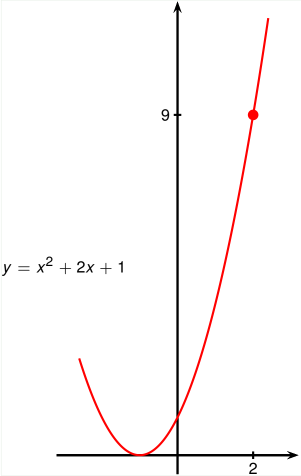
Find an equation for the tangent line to the parabola $y = x^2 + 2x + 1$ at the point $P = (2, 9)$.



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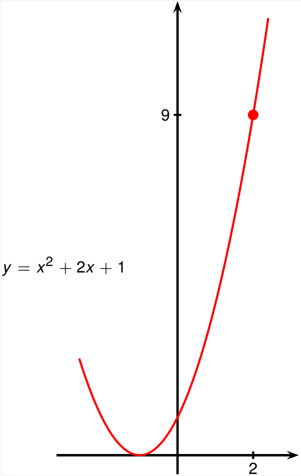

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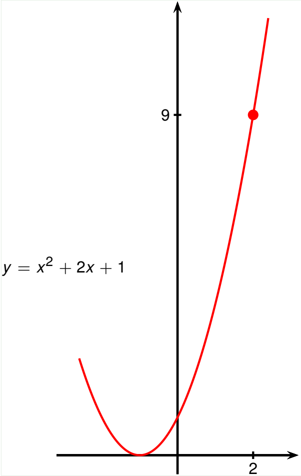

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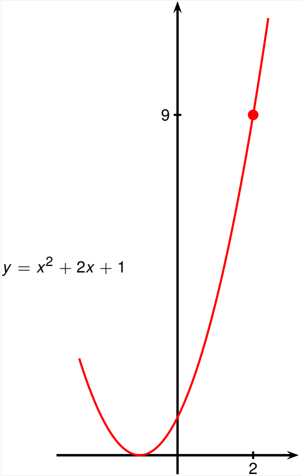

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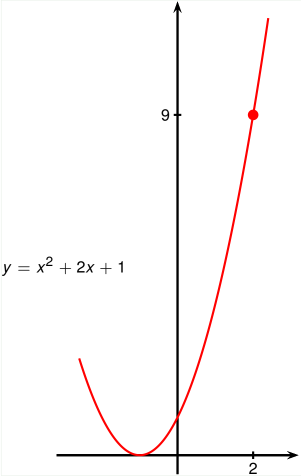

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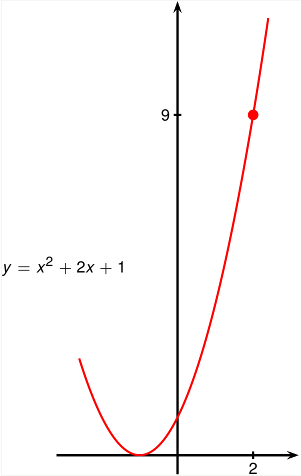

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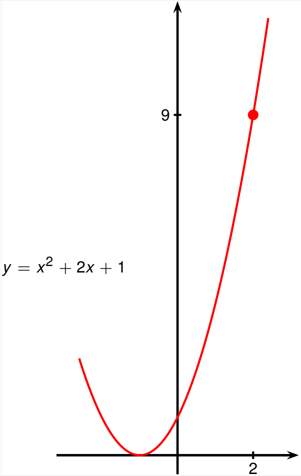

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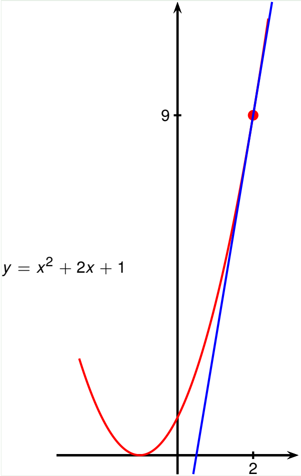

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The tangent line: $y = 6x - 3$.

Derivatives

Definition (Derivative)

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

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- Equivalent formulation. The derivative $f'(a)$ is the slope of the tangent line to $y = f(x)$ at $(a, f(a))$, provided that tangent line exists and is non-vertical.

Example

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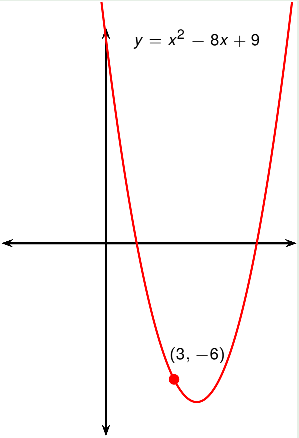

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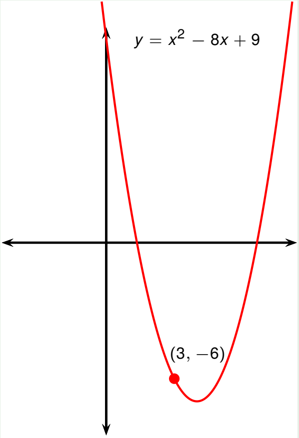

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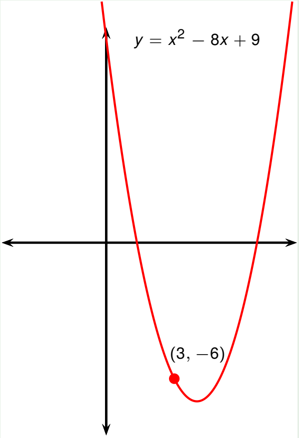

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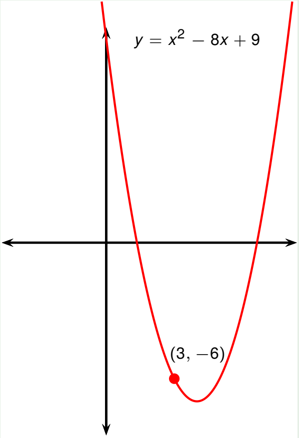

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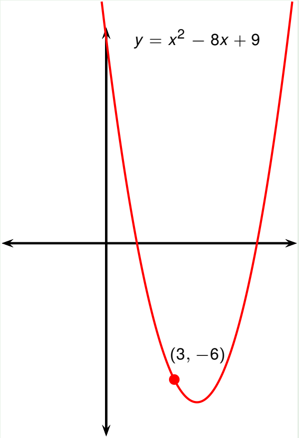

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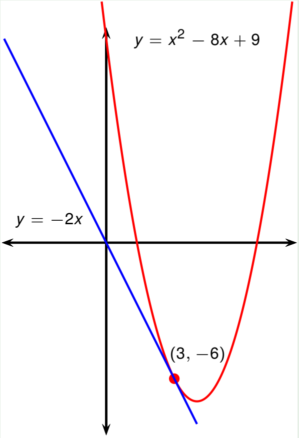

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- Therefore $f'(3) = 2 \cdot 3 - 8 = -2$.
- Point-slope form: $y - (-6) = -2(x - 3)$.
- Slope y-intercept form: $y = -2x$.

Other Notations

If $y = f(x)$ is a function, there are many ways to write its derivative.

$$f'(x) = y' = \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}f(x) = Df(x) = D_x f(x)$$

- D and $\mathrm{d}/\mathrm{d}x$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating the derivative.
- $\mathrm{d}y/\mathrm{d}x$ is called Leibniz notation, and should not be seen as a ratio; it just means the same as $f'(x)$.
- If we want to indicate the value of the derivative $\mathrm{d}y/\mathrm{d}x$ in Leibniz notation at a point a, we write

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_a \qquad \text{or} \qquad \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right]_a$$

The Derivative as a Function

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

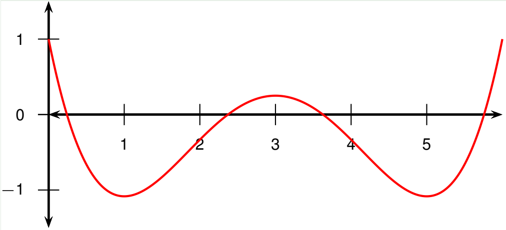
Now we change our point of view by letting the number a vary. If we replace the number a with the variable x, we get

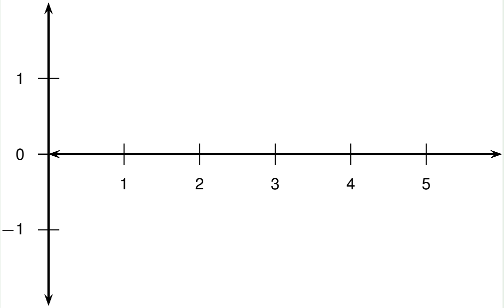
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

We regard f' as a new function, called the derivative of f.
The domain of f' is $\{x | f'(x) \text{ exists }\}$. It may be smaller than the domain of f.

Example

The graph of a function f appears below. Use it to sketch the graph of the derivative f'.





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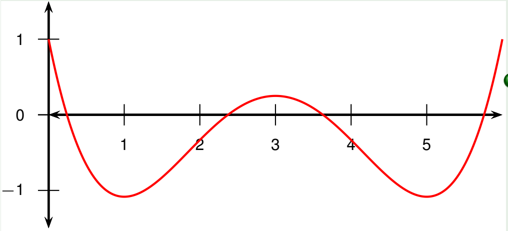

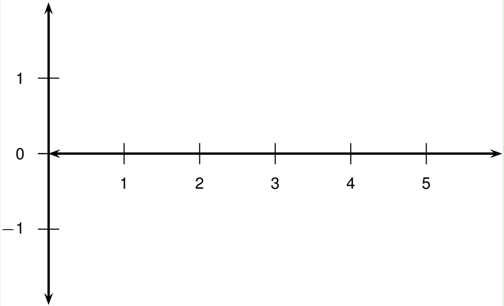

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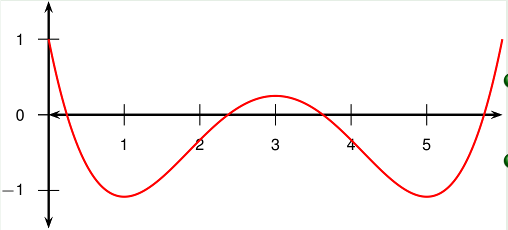

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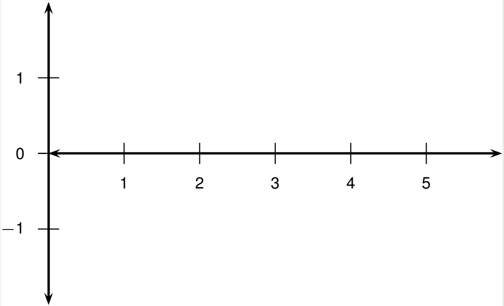

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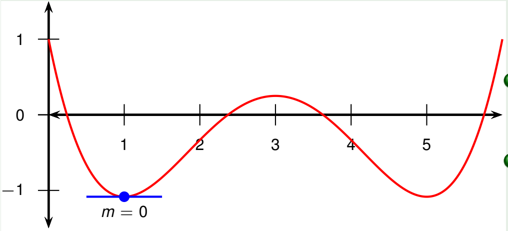

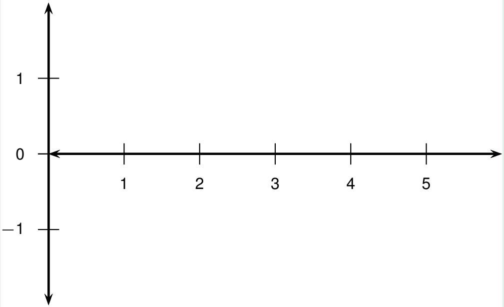

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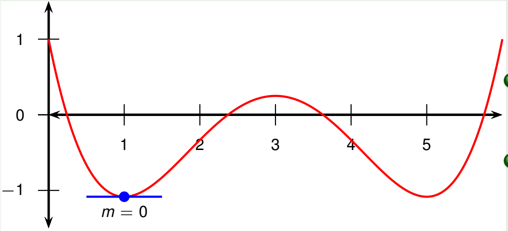

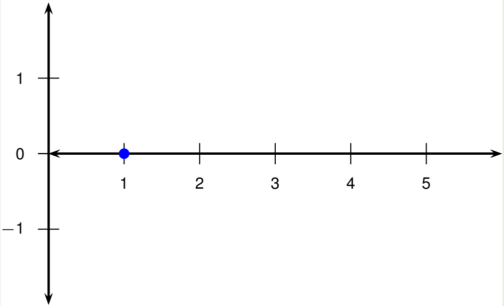

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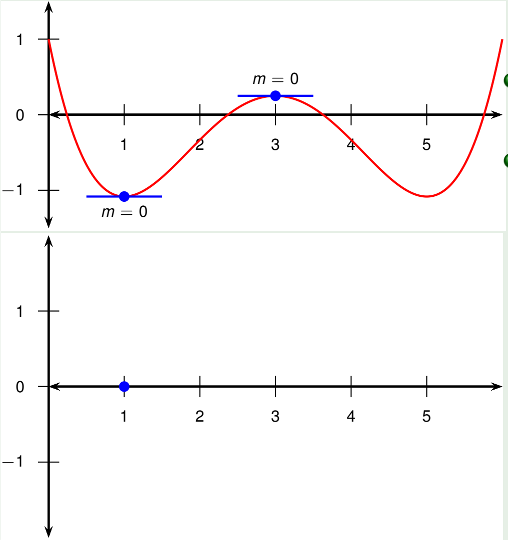

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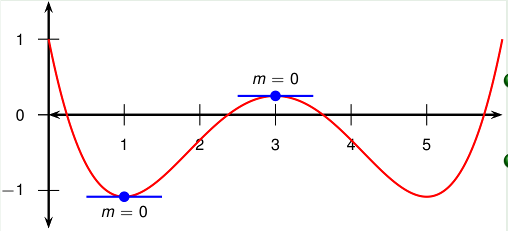

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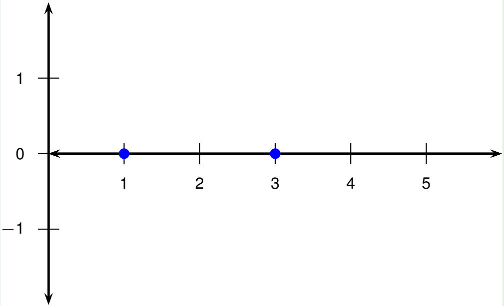
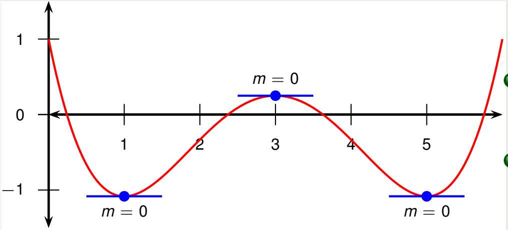
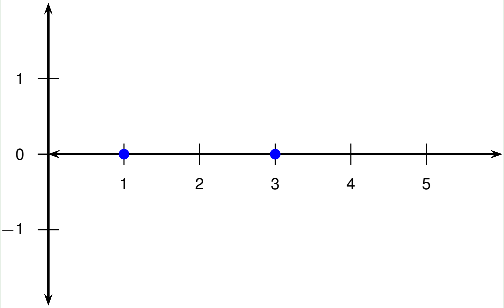


Example

The graph of a function f appears below. Use it to sketch the graph of the derivative f'.





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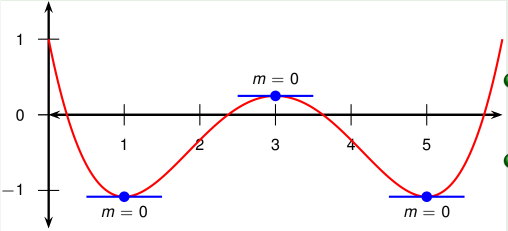

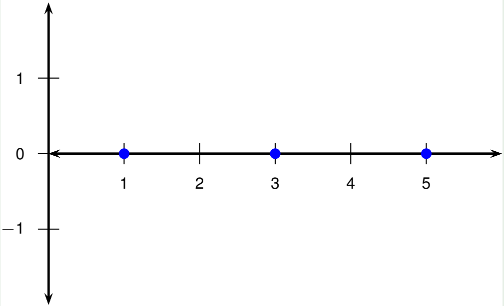

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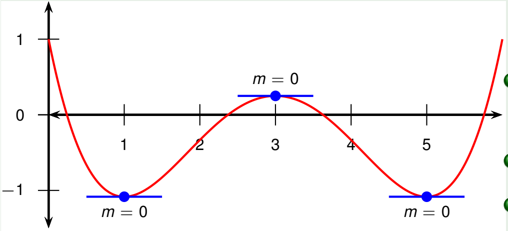

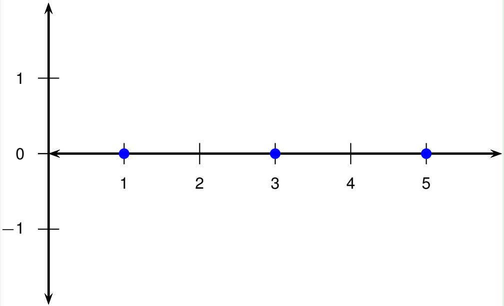

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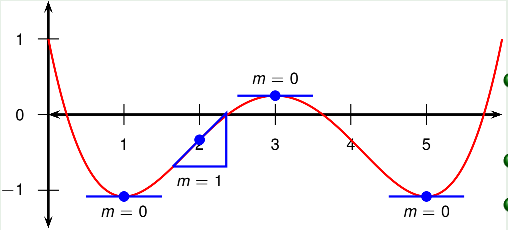

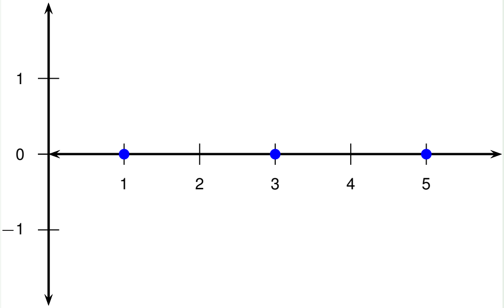

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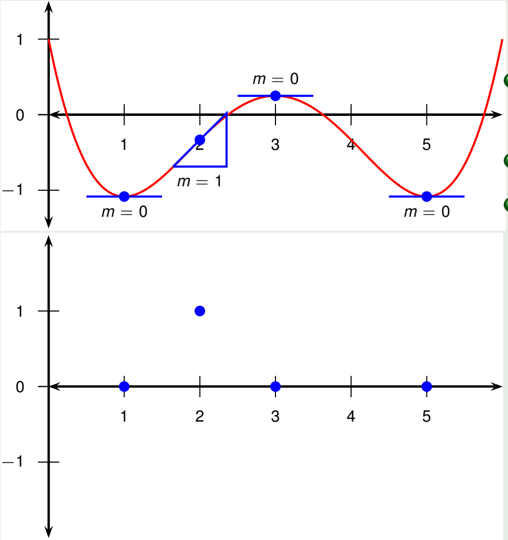

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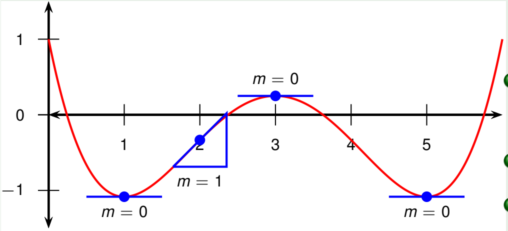

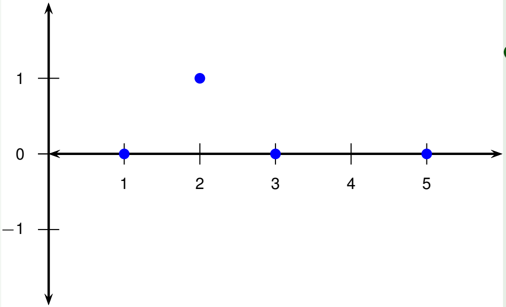

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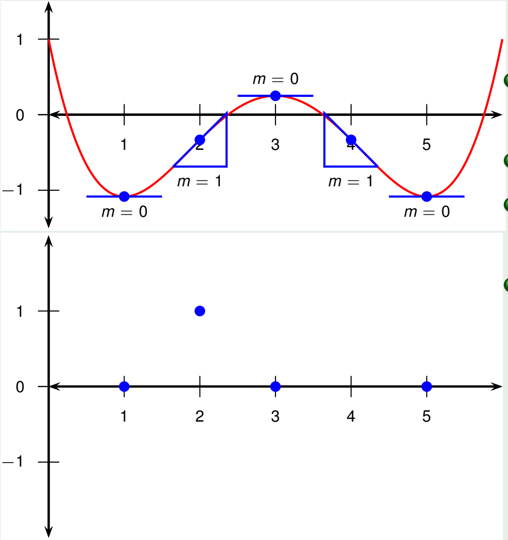

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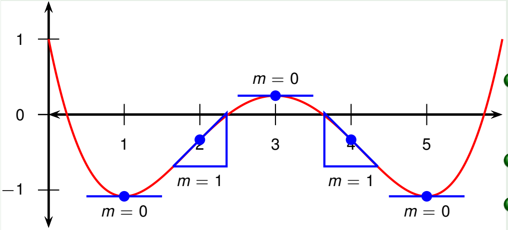

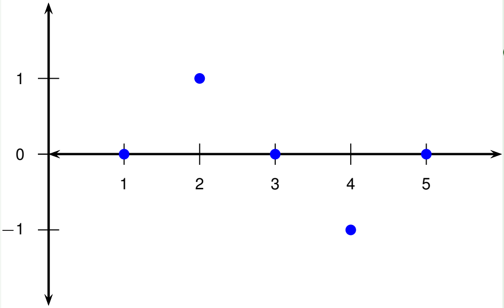

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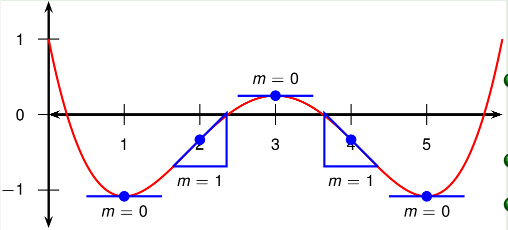

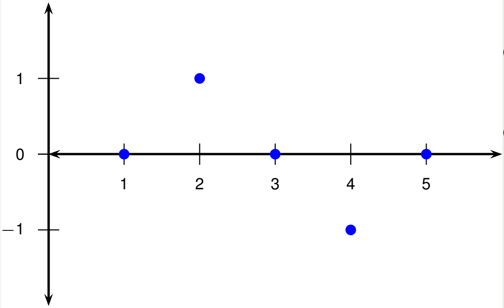

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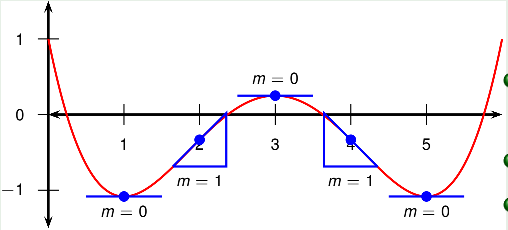

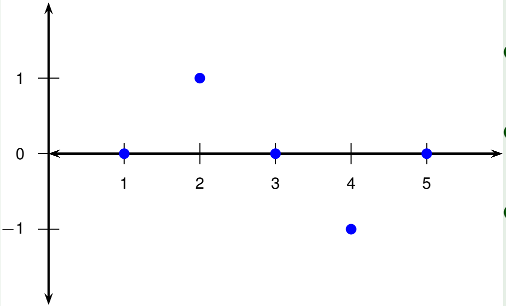

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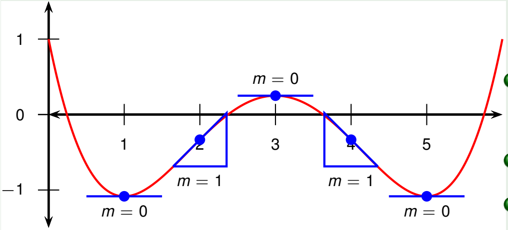

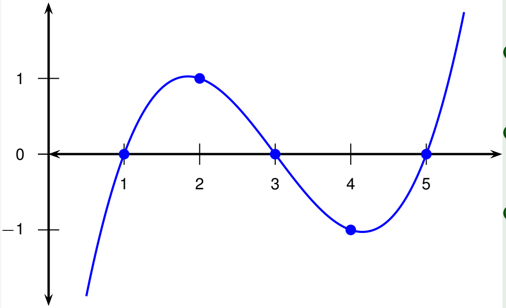

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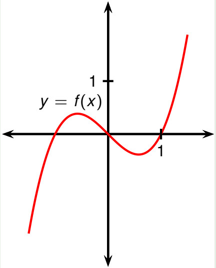
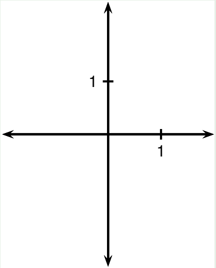
Example

If $f(x) = x^3 - x$, find the formula for $f'(x)$.





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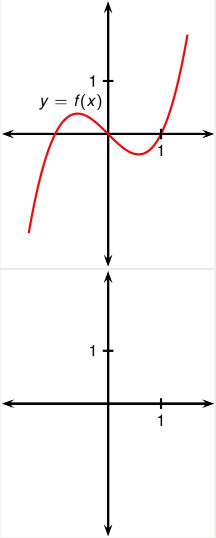

$$f'(x)$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

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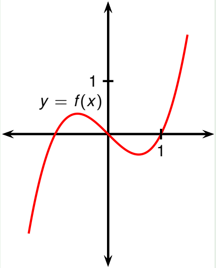

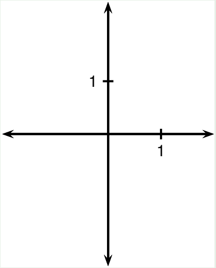

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f'(x) \\
&= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
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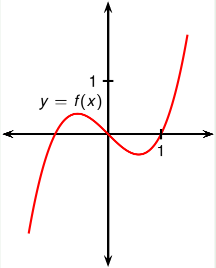

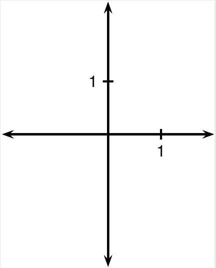

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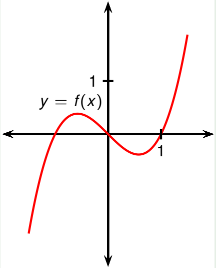

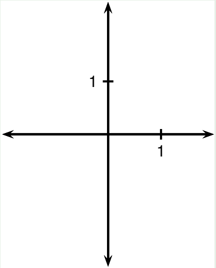

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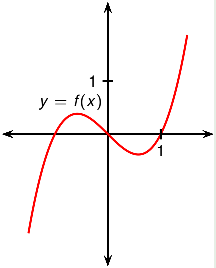

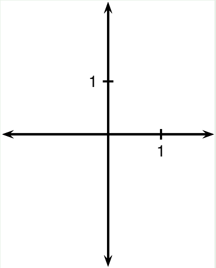

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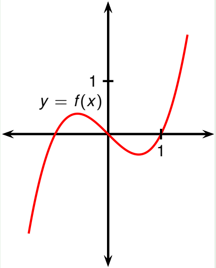

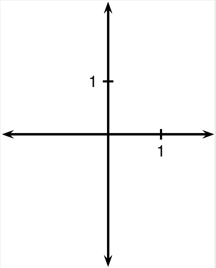

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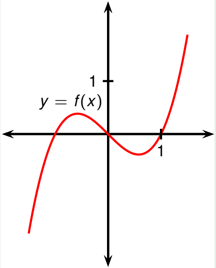

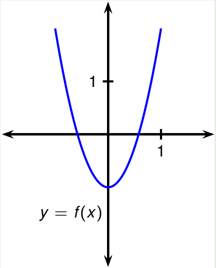

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