Solution to Problem Set 2

1. ①

$$\int \frac{dy}{y\cdot(10-y)} = \int dx \Rightarrow \int \frac{1}{10}\left(\frac{1}{y} + \frac{1}{10-y}\right)\cdot dy = \int dx \Rightarrow$$

$$\frac{1}{10}\left(\ln|y| - \ln|10-y|\right) = x + C \Rightarrow \ln\left|\frac{y}{10-y}\right| = 10x + C$$

$$\Rightarrow |y| = C\cdot e^{10x}\cdot|10-y|$$

lost solution: $y=0$ or $y=10$, ($y=0$ is included as above.)

②

$$\int \frac{dy}{y+1} = \int \frac{1}{x^2}\,dx \Rightarrow \ln|y+1| = \frac{-1}{x} + C \Rightarrow$$

$$|y+1| = C\cdot e^{-\frac{1}{x}}$$

lost solution: $y=-1$, included in above.

③

$$\int \frac{dy}{y^2+1} = \int \frac{1}{\sin x}\,dx \Rightarrow \arctan y = \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C$$

④

$$\frac{dy}{dx} + y = \sin x$$

$$m(x) = e^{\int 1\cdot dx} = e^{x+C} \quad \text{choose} \quad m(x) = e^x$$

$$y = \frac{1}{e^x}\cdot\int e^x\cdot\sin x\cdot dx = \frac{1}{e^x}\cdot\left[\frac{e^x\cdot(\sin x - \cos x)}{2} + c\right]$$

$$= \frac{\sin x - \cos x}{2} + c\cdot e^{-x}$$

⑤

$$y'' - y'\cdot\tan x = 1$$

let $y' = z$, $y'' = z'$, $\quad z' - z\cdot\tan x = 1 \quad m(x) = e^{-\int \tan x\,dx} = e^{\ln|\cos x| + C}$

choose $m(x) = \cos x$ then ~~to~~ $y' = z = \frac{1}{\cos x}\cdot\int \cos x\cdot 1\cdot dx = \tan x + \frac{C_1}{\cos x}$

$$\frac{dy}{dx} = \frac{\sin x + C_1}{\cos x} \quad \int dy = \int \frac{\sin x + C_1}{\cos x}\,dx \Rightarrow$$

$$y = -\frac{1}{2}\ln\cos^2 x + \frac{C_1}{2}\cdot\ln\frac{1-\sin x}{1+\sin x} + C_2$$

⑥. $\frac{dy}{dx} = -(1+3x^2)\cdot y \Rightarrow \int\frac{dy}{y} = \int -(1+3x^2)dx \Rightarrow$

$\ln|y| = -x - x^3 + C \qquad y = C\cdot e^{-x-x^3}$

lost solution: $y=0$. included already.

⑦. $\frac{dy}{dx} = \sqrt{y} \Rightarrow \int\frac{dy}{\sqrt{y}} = \int dx \Rightarrow 2\sqrt{y} = x + C. \Rightarrow$

$y = (\frac{x+C}{2})^2$, lost solution: $y=0$.

⑧ let $z(x) = x+y \qquad \frac{dz}{dx} = 1 + \frac{dy}{dx}$.

$\frac{dz}{dx} - 1 = \frac{dy}{dx} = z^2 \Rightarrow \frac{dz}{dx} = 1+z^2 \Rightarrow \int\frac{dz}{1+z^2} = \int dx$

$\arctan z = x + C \Rightarrow y + x = z = \tan(x+C) \Rightarrow y = \tan(x+C) - x$

1) X 2) ✓ 3) X 4) X 5) X 6) X

7) X 8) X

2. 1) $G(t)$: mass of additive in the tank.

$$\begin{cases} G'(t) = 40\times 2 - 45\cdot\underbrace{\frac{G(t)}{2000-5t}}_{} = 80 - \frac{45G(t)}{2000-5t} \\ G(0) = 100 \end{cases}$$

the volume of whole mixture as function of t.

$G'(t) + \frac{45}{2000-5t}\cdot G(t) = 80$

$m(t) = e^{\int\frac{45}{2000-5t}dt} = Ce^{-9\ln|2000-5t|}$

choose $m(t) = \frac{1}{(2000-5t)^9}$

$$G y(t) = (2000-5t)^9 \cdot \int \frac{80}{(2000-5t)^9} dt.$$

$$= (2000-5t)^9 \cdot \left[2 \cdot (2000-5t)^{-8} + c \right]$$

$$= 2 \cdot (2000-5t) + (2000-5t)^9 \cdot C$$

$$G y(0) = 2 \times 2000 + 2000^9 \cdot C = 100 \Rightarrow C = \frac{-3900}{2000^9}$$

$$G(t) = 2 \cdot (2000-5t) + \left(1 - \frac{t}{400}\right)^9 \cdot (-3900)$$

2).
$$\begin{cases} (66+7) \cdot \frac{dv}{dt} = -k \cdot v \\ v(0) = 9 \end{cases} \Rightarrow \int 73 \frac{dv}{v} = \int -k\,dt \Rightarrow$$

$$73 \cdot \ln|v| = -kt + C \Rightarrow v(t) = C \cdot e^{-\frac{3.9}{73}t}$$

$$v(0) = C = 9 \Rightarrow v(t) = 9 \cdot e^{-\frac{3.9}{73}t}$$

$v(t)$ will approach to 0 when $t \to \infty$.

a. $$s = \int_0^\infty v(t)\,dt. = \int_0^\infty 9 \cdot e^{-\frac{3.9}{73}t} dt = \lim_{A \to \infty} 9 \cdot (-e^{-\frac{3.9}{73}t}) / \frac{3.9}{73} \Big|_0^A = 9 \cdot \frac{73}{3.9} \text{ (km)}$$

b. $$v(t) = 9 \cdot e^{-\frac{3.9}{73}t} \leq 1 \iff -\frac{3.9}{73} t \leq \ln\frac{1}{9} \iff t \geq \frac{73 \cdot 2\ln 3}{3.9}$$

3)
$$\frac{dP(t)}{dt} = r \cdot P(t) \cdot (M_1 - P(t))$$

$$\int \frac{dP(t)}{P(t) \cdot (M_1 - P(t))} = \int r \cdot dt \Rightarrow \frac{1}{M_1} \cdot \ln\left|P(t)(M_1 - P(t))^{-1}\right| = rt + C$$

$$\Rightarrow P(t) \cdot (P(t) - M_1)^{-1} = C \cdot e^{M_1 r t}$$

$$P(0) \cdot (P(0) - M_1)^{-1} = C = P_0 \cdot (P_0 - M_1)$$

$$\Rightarrow P(t) \cdot (P(t) - M_1)^{-1} = P_0 \cdot (P_0 - M_1)^{-1} \cdot e^{M_1 r t}$$

$$1 + \frac{M_1}{P(t) - M_1} = \frac{P_0}{P_0 - M_1} \cdot e^{M_1 r t} \Rightarrow P(t) = \frac{M_1}{\frac{P_0}{P_0 - M_1} e^{M_1 r t} - 1} + M_1$$

4) $\frac{dX(t)}{dt} = X(t)\cdot[N - X(t)] \Rightarrow \int \frac{dX(t)}{X(t)\cdot[N-X(t)]} = \int dt \Rightarrow$

$$\frac{1}{N}\cdot\ln\left|\frac{X(t)}{N-X(t)}\right| = t + C \Rightarrow \frac{X(t)}{N-X(t)} = \tilde{C}\cdot e^{Nt}$$

3. 1) $\ln\left(\frac{1+t}{1-t}\right) = \ln(1+t) - \ln(1-t) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{(-1)^{n+1}\cdot x^n}{n} + o(x^n) -$

$[x + \frac{x^2}{2} + \frac{x^3}{3} + \dots + \frac{x^n}{n} + o(x^n)] = -x^2 - \dots - \frac{2x^{n}}{n} + o(x^n)$ for n even

$$T_n \ln\frac{1+t}{1-t} = \sum_{k=1}^{[\frac{n}{2}]} (-2)\cdot\frac{x^{2k}}{2k}$$

2) $\sin x\cdot\cos x = \frac{\sin 2x}{2} = \frac{1}{2}\left(2x - \frac{(2x)^3}{3!} + \dots + \frac{(-1)^{n+1}\cdot(2x)^{2n+1}}{(2n+1)!} + o(x^{2n+1})\right)$

$$T_{2n+1}\sin x\cdot\cos x = \frac{1}{2}\cdot 2x + \dots + \frac{(-1)^{n+1}(2x)^{2n+1}}{2\cdot(2n+1)!}$$

3) $\frac{1}{2-3x+x^2} = \frac{1}{1-x} - \frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = 1 + x + \dots + x^n + o(x^n) - \frac{1}{2}[1 + \frac{x}{2} + \dots$

$+ (\frac{x}{2})^n + o(x^n)] = \frac{1}{2} + \frac{3}{4}x + \dots + (1 - \frac{1}{2^{n+1}})\cdot x^n + o(x^n)$

$$T_n \frac{1}{2-3x+x^2} = \frac{1}{2} + \frac{3}{4}x + \dots + (1 - \frac{1}{2^{n+1}})\cdot x^n$$

4) $e^{t^2} = 1 + t^2 + \dots + \frac{(t^2)^n}{n!} + o(t^{2n})$

$$T_{2n} = 1 + t^2 + \dots + \frac{t^{2n}}{n!}$$

5) $\frac{\sin x}{x} = \frac{1}{x}\cdot\left(x - \frac{x^3}{3!} + \dots + \frac{(-1)^n x^{2n+1}}{(2n+1)!} + o(x^{2n+1})\right)$

$= 1 - \frac{x^2}{3!} + \dots + \frac{(-1)^n\cdot x^{2n}}{(2n+1)!} + o(x^{2n})$

$$T_{2n} = 1 + \frac{-1}{3!}x^{2\cdot 1} + \dots + \frac{(-1)^n}{(2n+1)!}\cdot x^{2n}$$

6). $[\arctan x]' = \dfrac{1}{1+x^2} = 1 - x^2 + (x^2)^2 + \cdots + (-1)^n \cdot (x^2)^n + o(x^{2n})$

$[T_{2n+1} \arctan x]' = T_{2n} \dfrac{1}{1+x^2} = 1 - x^2 + x^4 + \cdots + (-1)^n \cdot x^{2n}$

$T_{2n+1} \arctan x = C + x - \dfrac{x^3}{3} + \cdots + \dfrac{(-1)^n \cdot x^{2n+1}}{2n+1}$ $\quad C = \arctan(0) = 0.$

7) $\sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}(\sin x + \cos x) = \frac{\sqrt{2}}{2} \cdot \left(x - \frac{x^3}{3!} + \cdots + \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} + o(x^{2n+1}) \right)$

$+ \frac{\sqrt{2}}{2}\left(1 - \frac{x^2}{2!} + \cdots + \frac{(-1)^n \cdot x^{2n}}{(2n)!} + o(x^{2n}) \right)$

$= \frac{\sqrt{2}}{2} \cdot \left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n \cdot x^{2n}}{(2n)!} + o(x^{2n}) \right)$

$T_{2n} \sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2} \left\{ 1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n \cdot x^{2n}}{(2n)!} \right]$

$T_{2n+1} \sin(x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2} \left(1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n \cdot x^{2n}}{(2n)!} + \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} \right.$.

8) $[T_n \arcsin x]' = T_{n-1} \cdot \dfrac{1}{\sqrt{1-x^2}}$

$T_n \dfrac{1}{\sqrt{1-t}} = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} (-t)^k = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \cdot (-1)^k \cdot t^k.$

$T_{2n} \dfrac{1}{\sqrt{1-x^2}} = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} (-x^2)^k = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} (-1)^k \cdot x^{2k}$

$T_{2n+1} \arcsin x = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \cdot \dfrac{x^{2k+1}}{2k+1} \cdot (-1)^k$

9). $(2t+1)^{\frac{2}{3}}$

$T_n (1+x)^{\frac{2}{3}} = \sum_{k=0}^{n} \binom{\frac{2}{3}}{k} \cdot x^k$

$T_n (1+2t)^{\frac{2}{3}} = \sum_{k=0}^{n} \binom{\frac{2}{3}}{k} \cdot 2^k \cdot x^k$

4\. 1) $o(x^2) - o(x^2) = o(x^2)$

2) $o(x^2)\cdot(x^2-1) = o(x^2)$

3) $o(x) - o(x^3) = o(x)$

4) $\sin x^2 \cdot o(x) = o(x^3)$

5) $(e^x-1)\cdot o(x^2) = o(x^3)$

6) $\sin x \cdot \cos x = o(1)$

7) $3\cdot o(x) = o(x)$

8) $\sqrt{1+x} - \sqrt{1-x} = \dfrac{2x}{\sqrt{1+x}+\sqrt{1-x}} = o(1)$

5\. 1) $[(x+1)^{\frac{1}{2}}]^{(n)} = \binom{\frac{1}{2}}{n}\cdot n!\cdot(x+1)^{\frac{1}{2}-n}$

$$R_n f(x) = \frac{\binom{\frac{1}{2}}{n+1}\cdot(n+1)!\,(\xi+1)^{\frac{1}{2}-n-1}\cdot x^{n+1}}{(n+1)!} = \binom{\frac{1}{2}}{n+1}\cdot(1+\xi)^{-n-\frac{1}{2}}, \quad 0<\xi<x$$

$0<x<1 \Rightarrow \quad |R_n f(x)| \le \left|\binom{\frac{1}{2}}{n+1}\right|\cdot 1 = \left|\binom{\frac{1}{2}}{n+1}\right|$

2) $f(x)=e^x \qquad R_n f(x) = \dfrac{e^{\xi}\cdot x^{n+1}}{(n+1)!}$, $0<\xi<x$, $0\le x\le 1 \Rightarrow$

$$|R_n f(x)| \le \frac{e}{(n+1)!}$$

3) $f(x) = \sin x^2 \qquad R_n g(x) = \dfrac{g^{(n+1)}(\xi)\cdot x^{n+1}}{(n+1)!}$

$g(x) = \sin x \qquad R_{2n} f(x) = R_n g(x^2) = \dfrac{g^{(n+1)}(\xi)\cdot(x^2)^{n+1}}{(n+1)!}$, $0<\xi<x^2$

$0\le x\le 1 \Rightarrow \quad |R_n g(x^2)| \le \dfrac{1}{(n+1)!} = |R_{2n} f(x)|$