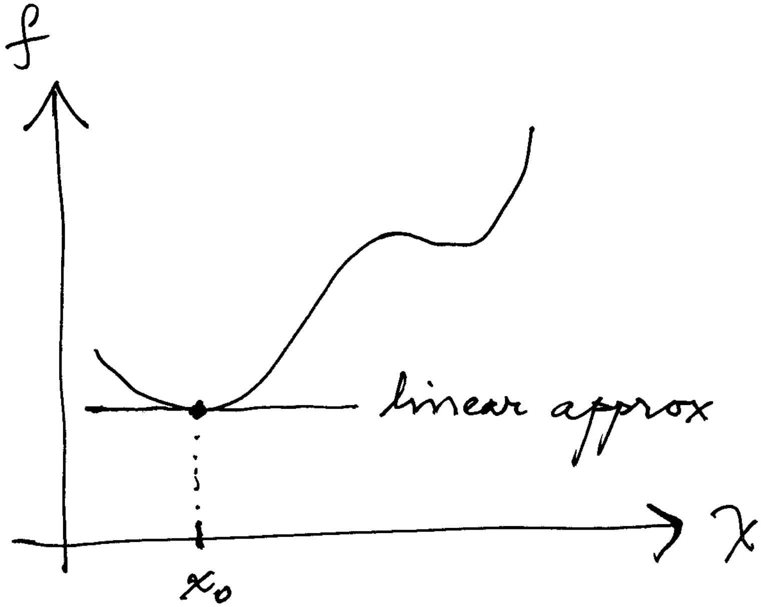
Approximating a function $f(x)$ with a polynomial



Requires $\tilde{f}(x_0) = f(x_0)$, $\tilde{f}'(x_0) = f'(x_0)$, $\tilde{f}''(x_0) = f''(x_0)$, etc.

F.ex.:

$$\tilde{f}(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \overset{x=x_0}{=} a_0$$

$$\tilde{f}'(x) = a_1 + 2a_2(x - x_0) \overset{x=x_0}{=} a_1$$

$$\tilde{f}''(x) = 2a_2$$

$$\tilde{f}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2$$

N-th order approximation:

$$\tilde{f}(x) = \sum_{n=0}^{N} a_n (x-x_0)^n \overset{x=x_0}{=} a_0 = f(x_0)$$

$$\tilde{f}'(x) = \sum_{n=1}^{N} n a_n (x-x_0)^{n-1} \overset{x=x_0}{=} a_1 = f'(x_0)$$

$$\tilde{f}''(x) = \sum_{n=2}^{N} n(n-1) a_n (x-x_0)^{n-2} \overset{x=x_0}{=} 2a_2 = f''(x_0)$$

$$\tilde{f}'''(x) = \sum_{n=3}^{N} n(n-1)(n-2) a_n (x-x_0)^{n-3} \overset{x=x_0}{=} 6a_3 = f'''(x_0)$$

$$\tilde{f}^{(n)}(x) = n!\, a_n = f^{(n)}(x_0) \rightarrow a_n = \frac{f^{(n)}(x_0)}{n!}$$

$$\therefore \quad \tilde{f}(x) = \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

If f is infinitely differentiable,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

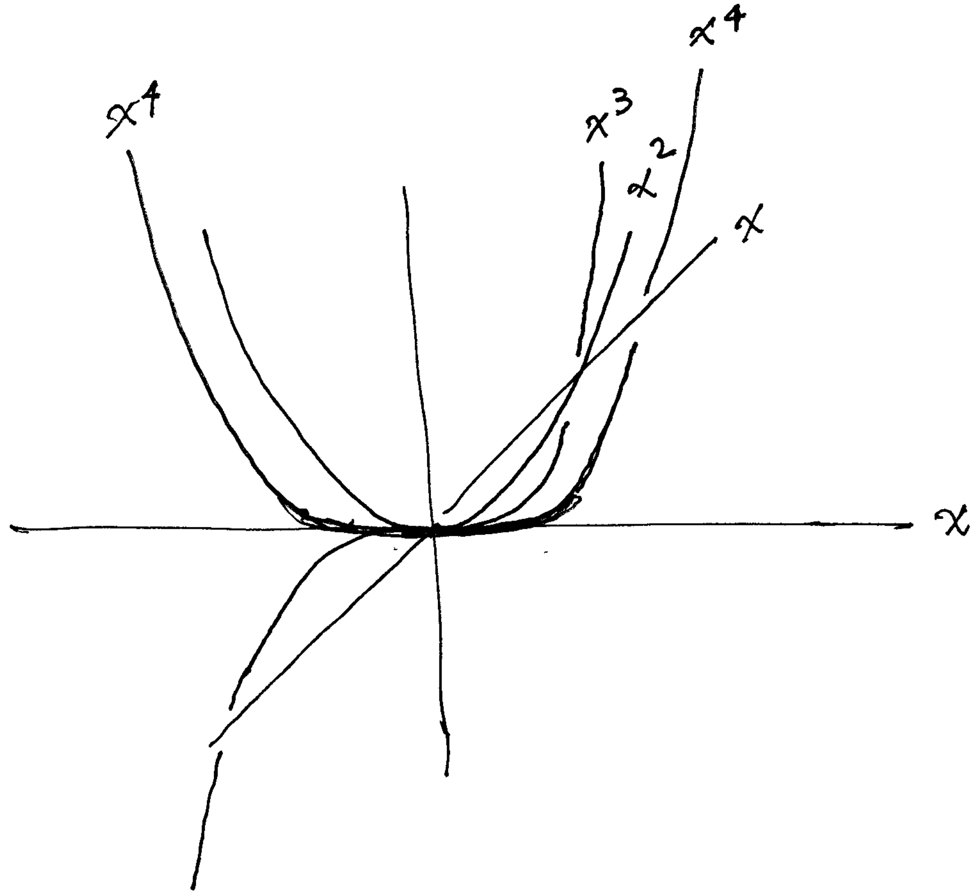
is the Taylor series expansion of $f(x)$

$$\tilde{f}_N(x) = \sum_{n=0}^{N} \left(\qquad \right) \quad \text{and} \quad f(x) = \sum_{n=0}^{\infty} \left(\qquad \right)$$

$$R_N(x) = f(x) - \tilde{f}_N(x) = \sum_{n=N+1}^{\infty} \left(\frac{f^{(n)}(x_0)}{n!} (x-x_0)^n \right)$$

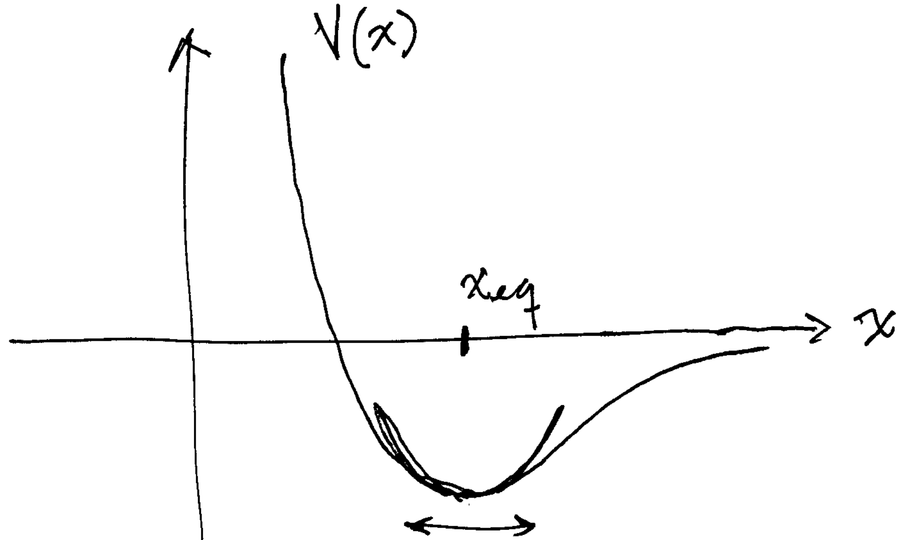
$$= (x-x_0)^{N+1} \underbrace{\sum_{n=N+1}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x-x_0)^{n-(N+1)}}_{\sum_{m=0}^{\infty} a_{m+N+1} (x-x_0)^m}$$

As $x \to x_0$ $\quad R_N(x) \to (x-x_0)^{N+1} \cdot a_{N+1}$



Frequent terminology : $R_N = O(x^{N+1})$

Application :



2^{nd} order approx. is good
$\rightarrow$ SHO.