Critical damping: $q=0$, $\gamma = \omega_0$

$$[D^2 + 2\gamma D + \omega_0^2]x = 0 \quad \rightarrow \quad [(D+\gamma)^2]x = 0$$

i.e. $$[D+\gamma]\underbrace{\left([D+\gamma]x\right)}_{\text{call this } y(t)} = 0$$

Then $[D+\gamma]y = 0 \quad \rightarrow \quad y(t) = Ae^{-\gamma t}$

and $[D+\gamma]x = Ae^{-\gamma t}$

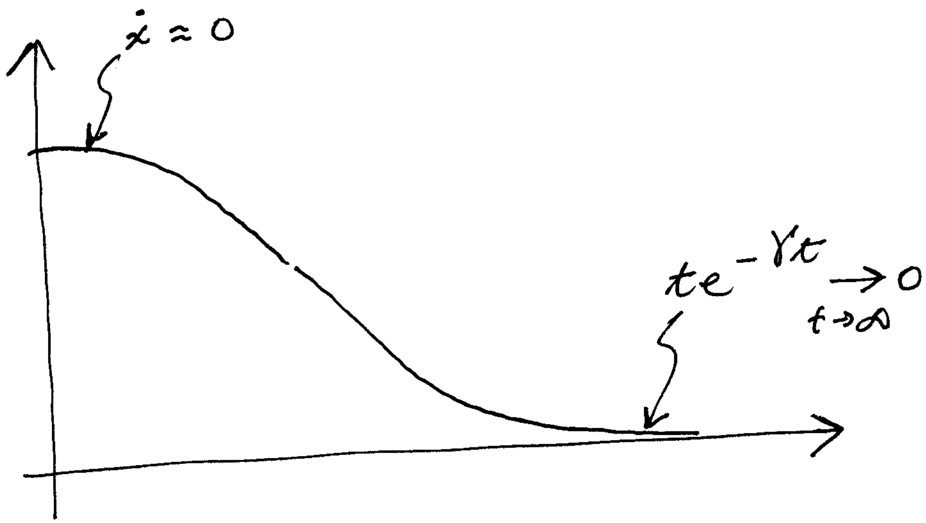
Dirty trick: $A = e^{\gamma t}[D+\gamma]x = D(xe^{\gamma t})$

$$At + B = xe^{\gamma t}$$

$$\boxed{x = e^{-\gamma t}(At+B)}$$

E.g. $x = x_0$, $\dot{x} = 0$ at $t = 0$

$$\longrightarrow \left.\begin{aligned} 0 &= -\gamma B + A \\ x_0 &= B \end{aligned}\right\} \longrightarrow x = x_0(\gamma t + 1)e^{-\gamma t}$$



Overdamping: $\gamma > \omega_0$

Again $\quad x(t) = e^{-\gamma t}\left(Ae^{qt} + Be^{-qt}\right)$

but now $\quad q = \sqrt{\gamma^2 - \omega_0^2}$ is real.

E.g. $\quad x = x_0\,,\ \dot{x} = 0$ at $t = 0$

$$\longrightarrow \left.\begin{aligned} x_0 &= A + B \\ -\frac{\gamma x_0}{q} &= A - B \end{aligned}\right\} \longrightarrow \begin{aligned} A &= x_0\,\frac{q-\gamma}{2q} \\ B &= x_0\,\frac{q+\gamma}{2q} \end{aligned}$$

$$x = \frac{x_0}{2q}\, e^{-\gamma t}\left((q-\gamma)e^{qt} + (q+\gamma)e^{-\gamma t}\right)$$

$$x \xrightarrow[t\to\infty]{} \frac{x_0}{2q}(q-\gamma)e^{(q-\gamma)t} \longrightarrow 0 \quad \text{because } q<\gamma$$

So again :