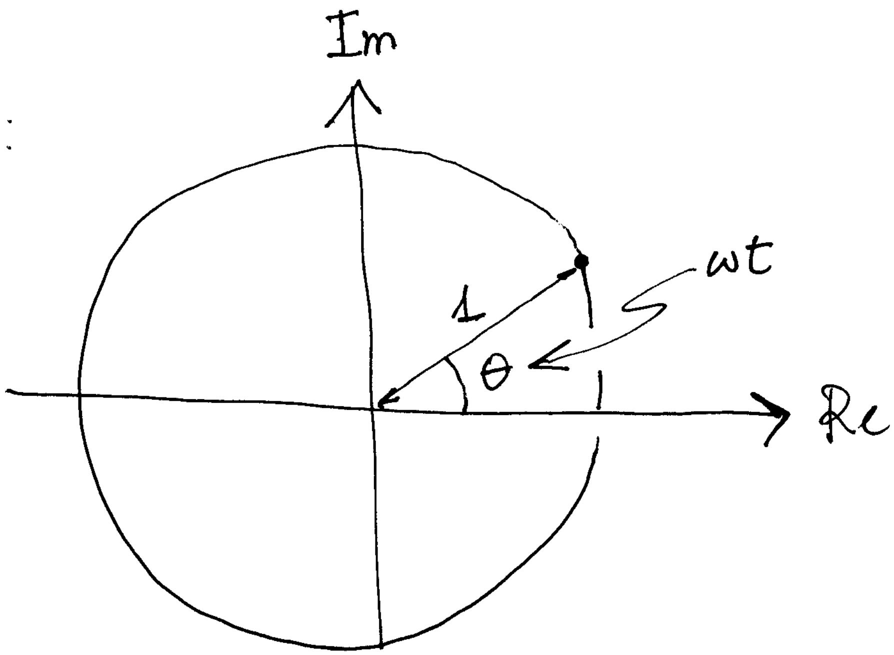
Why use complex numbers?

F. ex SHO $\ddot{x} + \omega^2 x = 0$

Solve $\ddot{\tilde{x}} + \omega^2 \tilde{x} = 0$

Try $\tilde{x} = A e^{i(\omega t + \phi_0)}$ works!

$e^{i\theta}$:



Back to the forced oscillator:

$$\ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

$$\longrightarrow \quad \ddot{\tilde{x}} + \omega_0^2 \tilde{x} = \frac{F_0}{m} e^{i\omega t}$$

Steady state : Try $\tilde{x} = A e^{i(\omega t - \phi)}$

$$\dot{\tilde{x}} = i\omega A e^{i(\omega t - \phi)}$$

$$\ddot{x} = -\omega^2 A e^{i(\omega t - \phi)}$$

$$(\omega_0^2 - \omega^2) A e^{i(\cancel{\omega t} - \phi)} = \frac{F_0}{m} \cancel{e^{i\omega t}}$$

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m} e^{i\phi}$$

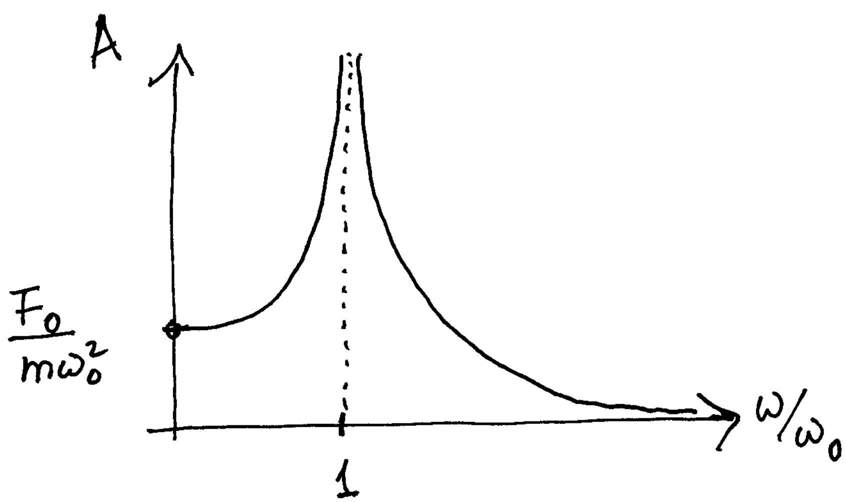
$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m} \cos\phi$$

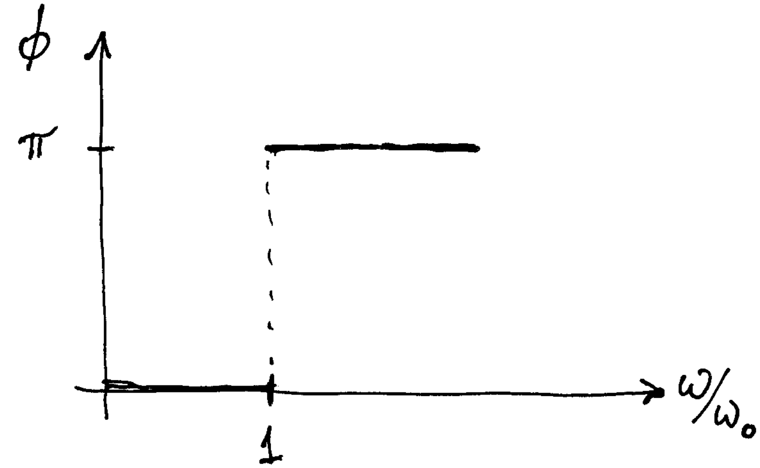
$$0 = \frac{F_0}{m} \sin\phi$$

$\sin\phi = 0 \rightarrow \phi = 0$ or π

$\phi = 0 \rightarrow \cos\phi = 1 \rightarrow A = \dfrac{F_0/m}{\omega_0^2 - \omega^2} \leftrightarrow \omega < \omega_0$

$\phi = \pi \rightarrow \cos\phi = -1 \rightarrow A = \dfrac{F_0/m}{\omega^2 - \omega_0^2} \leftrightarrow \omega > \omega_0$





DAMPED, FORCED OSCILLATION:

becomes $e^{i\omega t}$

$$\ddot{\tilde{x}} + 2\gamma\dot{\tilde{x}} + \omega_0^2\tilde{x} = \frac{F_0}{m}\cos(\omega t)$$

Try $\quad \tilde{x} = A e^{i(\omega t - \phi)}$

$$(\omega_0^2 - \omega^2 + 2i\gamma\omega) A e^{i(\cancel{\omega t}-\phi)} = \frac{F_0}{m}\cancel{e^{i\omega t}}$$

$$(\omega_0^2 - \omega^2) A = \frac{F_0}{m}\cos\phi$$

$$2\gamma\omega A = \frac{F_0}{m}\sin\phi$$

$$\tan\phi = \frac{2\gamma\omega}{\omega_0^2 - \omega^2}$$

$$A = \frac{F_0/m}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}$$