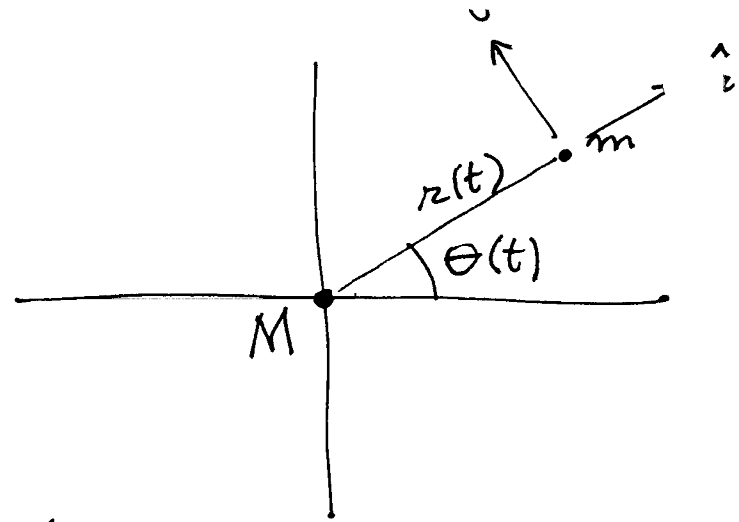
EOM: $\cancel{m}\ddot{\vec{r}} = -G\frac{\cancel{m}M}{r^2}\hat{r}$



The angular momentum

$\vec{L} = \vec{r} \times m\vec{v}$ is conserved.

In polar coordinates, $\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$

$$\therefore \quad \boxed{\ddot{r} - r\dot{\theta}^2 = -G\frac{M}{r^2}} \leftarrow \text{equation to solve}$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

$$\frac{d}{dt}\frac{|\vec{L}|}{m}$$

$$\vec{L} = m\vec{r} \times \vec{v} = mr^2\dot{\theta}\,\hat{k}$$

$r\hat{r}$ $\qquad$ $\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$$\vec{L} = \underbrace{mr^2}_{I}\;\underbrace{\dot{\theta}\,\hat{k}}_{\vec{\omega}}$$

$\vec{\omega}$: angular velocity

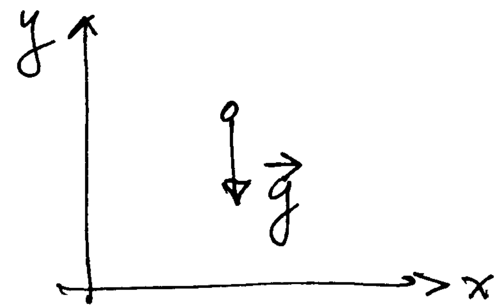
I : moment of inertia

$\vec{L} = I\vec{\omega}$, analogous to $\vec{p} = m\vec{v}$

We just want the trajectory, $r(\theta)$ instead of $r(t)$ and $\theta(t)$.

Simpler, but similar problem: Projectile

$$\frac{d^2y}{dt^2} = -g$$



$$\frac{d^2x}{dt^2} = 0 \rightarrow \frac{dx}{dt} = v_0 = \text{constant}$$

Then $$\boxed{\frac{d*}{dx}} = \frac{d*}{dt} \Big/ \frac{dx}{dt} = \boxed{\frac{1}{v_0} \cdot \frac{d*}{dt}}$$

and $$\frac{d*}{dt} = v_0 \frac{d*}{dx}$$

$$\frac{dy}{dt} = v_0 \frac{dy}{dx} \quad \text{and} \quad \frac{d^2y}{dt^2} = v_0^2 \frac{d^2y}{dx^2}$$

Then $$v_0^2 \frac{d^2y}{dx^2} = -g \quad \text{or} \quad \boxed{\frac{d^2y}{dx^2} = -\frac{g}{v_0^2}}$$

$$\rightarrow \quad \frac{dy}{dx} = -\frac{g}{v_0^2} x + A$$

and $$y = -\frac{g}{2v_0^2} x^2 + Ax + B$$

Back to the 2BP: $mr^2\dot{\theta}$ is conserved,

define $$r^2\dot{\theta} = \frac{L}{m} = l$$

Then $$r^2\frac{d\theta}{dt} = l \rightarrow \frac{d\theta}{dt} = \frac{l}{r^2} \rightarrow \frac{d*}{dt} = \frac{l}{r^2}\frac{d*}{d\theta}$$

Convenient change of variable: $u = \frac{1}{r}$

Then $$\frac{d\theta}{dt} = lu^2 \rightarrow \frac{d*}{dt} = lu^2\frac{d*}{d\theta}$$

$$\frac{du}{dt} = lu^2\frac{du}{d\theta} \quad \text{and} \quad \frac{d^2u}{dt^2} = \frac{d}{dt}\left(lu^2\frac{du}{d\theta}\right)$$

$$= lu^2\frac{d}{d\theta}\left(lu^2\frac{du}{d\theta}\right)$$

$$= l^2\left(2u^3\frac{du}{d\theta} + u^4\frac{d^2u}{d\theta^2}\right)$$

$$\frac{dr}{dt} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\left(l u^2 \frac{du}{d\theta}\right) = -l\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2} = \cdots = -u^2 l^2 \frac{d^2u}{d\theta^2}$$

The EOM becomes

$$-u^2l^2\frac{d^2u}{d\theta^2} - u^3l^2 = -GMu^2$$

$$\rightarrow \boxed{\frac{d^2u}{d\theta^2} + u = \frac{GM}{l^2}}$$

This is a SHO! $u(\theta) = \frac{GM}{l^2} + A\cos(\theta + \theta_0)$

choose $\theta_0 = 0$