Lecture 4 More Graphs

Lecturer: Anup Rao Scribe:

1 Mantel's Theorem

Theorem 1. *If a graph on n vertices has $n^2 + 1$ edges, then it has a triangle.*

Proof We prove it by induction on n. When $n = 1$, the theorem is true, since the number of edges is at most $1 < n^2 + 1$.

In the general case, suppose the graph G has $2(n + 1)$ vertices. Let $\{xy\}$ be an edge in the graph. Consider the graph G' on $2n$ vertices obtained by deleting x, y from the original graph. If G' has at least $n^2 + 1$ edges, then it has a triangle by induction, and we are done. Otherwise, there must be $(n + 1)^2 + 1 - n^2 - 1 = 2n + 1$ edges that connect x, y to the vertices of G'. Thus by the pigeonhole principle, there is some vertex z so that $\{x, z\}, \{y, z\}$ are both edges. Then x, y, z form a triangle. ■

The above theorem is tight. Consider the graph with n vertices on the left and n vertices on the right and every vertex on the left is connected to every vertex on the right. This graph has no triangles but n^2 edges.

2 In-class Exercise

1. Show that the number of people that have an odd number of friends is even. *Solution*: Last class we showed that in any graph with n vertices and m edges, we have that $2m = \sum_v degree(v)$. If there are an odd number of edges with odd degree, then the sum on the right becomes odd, which is impossible.

2. Show that if a graph does not have any cycles, then it has at most $2n/3$ vertices whose degree is at least 3. *Solution*: Suppose we have a graph with $2n/3$ vertices that have degree at least 3. Then $\sum_v degree(v) \geq (2n/3)3 = 2n$. Therefore by the formula above, the graph has n edges, and we have proved on the homework that the graph must have a cycle.

3. A planar graph is a graph that can be drawn in the plane without crossing any edges. For planar graphs it is always true that $m \leq 3n - 6$, as long as $n \geq 3$. Show that you can always color the vertices of a planar graph using 6 colors, in such a way that every edge gets two distinct colors. *Solution*: Next time.