Home Work (supplemental to class work 7)
Alphabet Reduction through composition

due: January 1, 2014

The last remaining part of the proof of the PCP theorem is alphabet reduction. Towards this we will first show how to robustize a constraint graph.

1 Robust value of a constraint graph

Let $G = (V, E, \Pi)$ be a constraint graph over alphabet $\Sigma = \{0,1\}^{\ell}$, $\Pi = \{\pi_{uv}\}_{uv \in E}$. Recall that the unsat value of G is the minimum over all possible assignments $a : V \to \Sigma$ of $\Pr_{uv \in E}[(a(u), a(v)) \notin \pi_{uv}]$, the fraction of constraints violated by a.

For a constraint $\pi \subset \Sigma \times \Sigma = \{0,1\}^{2\ell}$, we define

Definition 1.1. Let $0 < \rho < 1/2$. We say that the constraint π is ρ-unsatisfied by a string $s \in \{0,1\}^{2\ell}$ if every $a \in \pi \subset \{0,1\}^{2\ell}$ differs from s on at least $\rho \cdot 2\ell$ coordinates.

Definition 1.2 (ρ-robust soundness)**.** The ρ-robust soundness of a constraint graph, denoted $\text{unsat}_{\rho}(G)$, is the minimum over all assignments $a : V \to \Sigma$ of the fraction of ρ-unsatisfied constraints.

Question 1. Find a linear time algorithm that inputs a constraint graph $G = (V, E, \Pi)$ over constant-size alphabet $\Sigma = \{0,1\}^{\ell}$ and outputs a constraint graph $G' = (V, E, \Pi')$ over alphabet $\Sigma = \{0,1\}^{2\ell}$ such that, for some small constant $\rho > 0$, $\text{unsat}_{\rho}(G') = \text{unsat}(G)$.

(Hint: use the existence of an error correcting code $C : \{0,1\}^{\ell} \to \{0,1\}^{2\ell}$ that guarantees that for every $x \neq y \in \{0,1\}^{\ell}$, $dist(C(x), C(y)) > \delta \cdot 2\ell$ for some constant $\delta > 0$. This implies that for every $z \in \{0,1\}^{2\ell}$ there is at most one codeword $C(x)$ within Hamming distance $\delta\ell$ of z.)

2 Assignment Tester from PCPP

Please read `http://www.tcs.tifr.res.in/ prahladh/teaching/07autumn/lectures/lec4.pdf`. In addition, please sove the question at the end of this section.

The main theorem proved in the reading is that

$$NP \subseteq PCP_{1,0.99}[poly(n), O(1)]$$

In the last subsection this is strengthened to give a PCPP (PCP of Proximity). Our goal is to use this theorem for alphabet reduction, and next week we will analyze how this is done. For now, we will show that a construction of a PCPP implies an assignment tester, defined next.

Definition 2.1 (Assignment tester)**.** A q-query Assignment Tester AT ($\delta > 0, \Sigma_0$) is a reduction algorithm P whose input is a Boolean circuit Φ over Boolean variables X and whose output is a system of constraints $\Psi = \{\psi_1, \ldots, \psi_m\}$ over X and additional variables Y such that

- The variables in Y take values in Σ_0
- Each $\psi \in \Psi$ depends on at most q variables in $X \cup Y$
- For every assignment $a : X \to \{0, 1\}$
  - Completeness: If $\varphi(a) = 1$, then there is an assignment $b : Y \to \Sigma_0$ such that $a \cup b$ satisfies all $\psi \in \Psi$.
  - Soundness: If a is δ-far from every assignment a' such that $\varphi(a') = 1$ then for every $b : Y \to \Sigma_0$, at least δ fraction $\psi \in \Psi$ are violated by $a \cup b$.

Question 2. Prove that there exists an $O(1)$-query assignment tester algorithm that inputs a circuit Φ of size n over variables X, and outputs a system of $m = \exp(poly(n))$ constraints, such that the above completeness and soundness conditions hold. You may, of course, use the results of the reading.