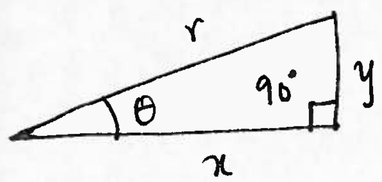
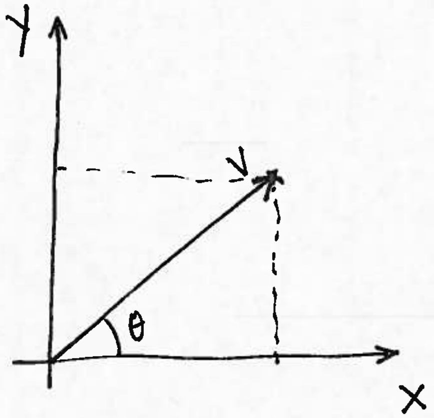
TRIGONOMETRY BASICS



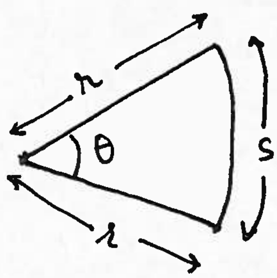
$$\sin\theta = \frac{y}{r} \; ; \; \cos\theta = \frac{x}{r}.$$



Let vector $\vec{V}$ be a vector in coordinate system XY. Let $\vec{V}$ make an angle θ wrt. x-axis at the origin at shown. The x-projection of $\vec{V}$ is $V_x = |V| . \cos\theta$.

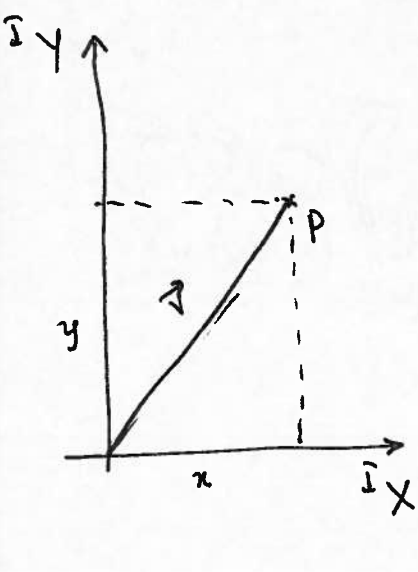
Y-projection $V_y = |V| . \sin\theta$.

where $|V|$ is the length of V.



Given the radius r, and an angle θ, the arc s formed has length $s = r . \theta$ where θ is in radians.

LINEAR ALGEBRA NOTATION



Coordinate System $\equiv ({}^{I}X\,{}^{I}Y)$

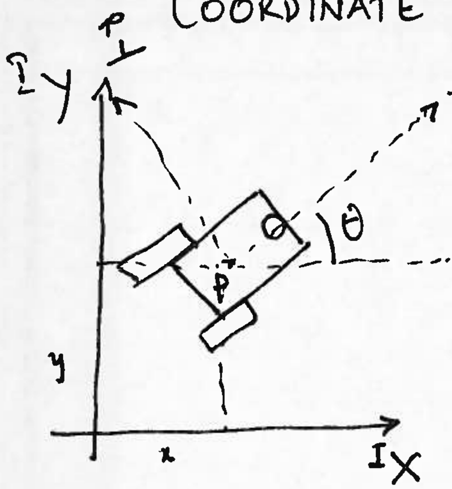
$\vec{V}$ = vector

${}^{I}P$ - Point P in coordinate system I.

$${}^{I}P = (x, y)$$

$$|\vec{V}| = \sqrt{x^2 + y^2}$$

COORDINATE SYSTEMS FOR WHEELED ROBOTS



$^{I}X\,^{I}Y \equiv$ Global coordinate system
or
inertial coordinate system

$^{R}X\,^{R}Y \equiv$ Local coordinate system
or
robot coordinate system

ξ (Greek letter Xi) $\equiv$ Robot pose.

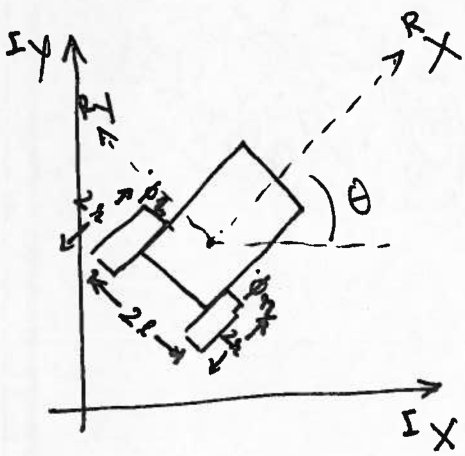
$$^{I}\xi = [\, x \;\; y \;\; \theta \,]$$

NOTE: As shown in figure, the X-axis of the local coordinate system is aligned with the robot's forward direction and the Y-axis is aligned with the perpendicular with the robot at the origin. ~~Therefore~~ $^{R}\xi = [0, 0, 0]$ always.

FORWARD KINEMATICS

Definition: Given the joint angles, we want to calculate the position and velocity of the robot in the inertial (global) frame.

For wheeled robots, the input is the robot geometry and the angular velocities of each of the wheels.

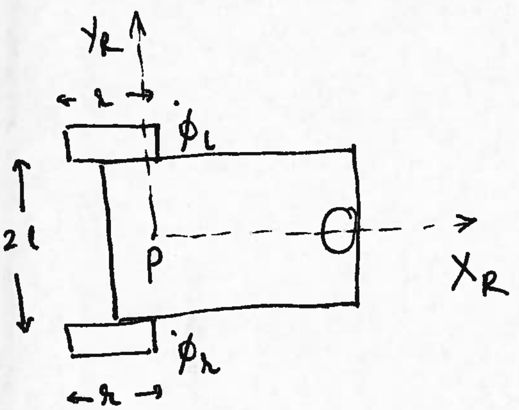


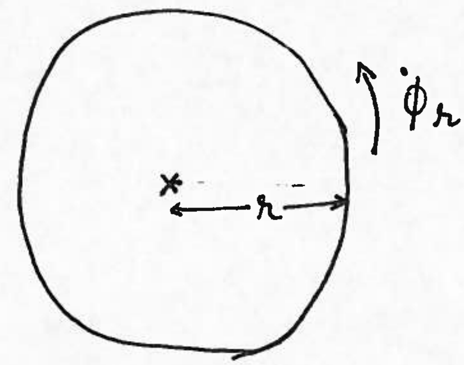
Let the wheels be equidistant from the center of gravity of the robot, each at a distance of 'l'. Let 'r' be the radius of both wheels. Let $\dot{\phi}_r$ and $\dot{\phi}_l$ be the angular velocities at which the right and left wheels are spinning respectively.

The objective of forward Kinematics of wheeled robot is to determine the function 'f' that maps robot velocities in inertial space to the angular velocities of the wheels

$$ {}^{I}\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(l, r, \theta, \dot{\phi}_l, \dot{\phi}_r) $$

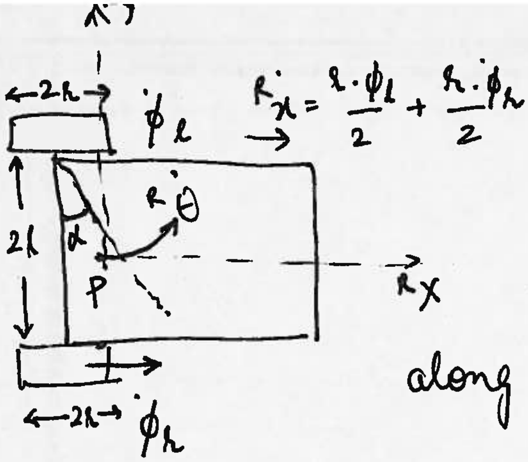
Step 1: Map angular velocities of the wheels to robot velocities in the robot's local frame.





Given the angular velocity of a wheel ($\dot{\phi}_r$ for the right wheel) it's linear velocity is $r \cdot \dot{\phi}_r$. Let us now assume that only right wheel is spinning. This would cause the robot to turn around the left wheel with the right wheel traveling at a speed of $r \cdot \dot{\phi}_r$. This causes the robot to experience a velocity of $\frac{r \cdot \dot{\phi}_r}{2}$ at its center of gravity P.

Similarly, if only the left wheel was turning, the robot would experience a velocity of $\frac{r \cdot \dot{\phi}_l}{2}$ at its center of gravity. Therefore, the robots velocity along the local x-axis is $\quad {}^{R}\dot{x} = \frac{r \cdot \dot{\phi}_r}{2} + \frac{r \cdot \dot{\phi}_l}{2}$.



Assuming that there is no slippage, the instantaneous velocity of the robot in the local y-direction is zero because the wheels are aligned along the local x-axis and fixed.

$$\cancel{{}^R\dot{y}=0}. \quad {}^R\dot{y} = 0.$$

Let us next compute the angular velocity ${}^R\dot\theta$ in the local coordinate frame. Let us assume that angular rotation in the counter-clockwise direction is positive and in the clockwise direction is negative.

Again, let us assume that only the right wheel is spinning. It makes the robot turn counter-clockwise around the left wheel at a linear speed of $\dot\phi_r \cdot r$. ~~The counter-clockw~~ The angular velocity (labeled α) is

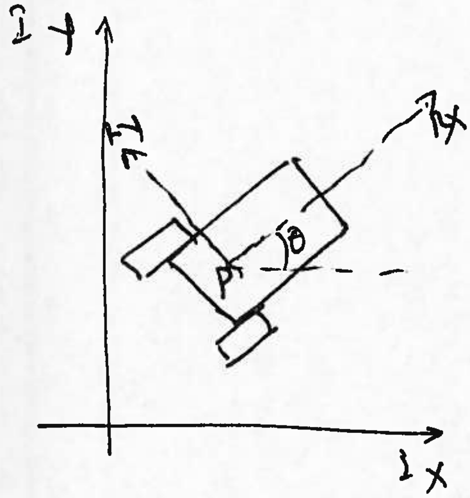
$$\alpha = \frac{\dot\phi_r \cdot r}{2l}$$

Similarly, the angular velocity caused by the left wheel spinning is $-\frac{\dot\phi_l \cdot r}{2l}$. It is negative because it causes the robot to spin in the clockwise direction. Therefore, the angular velocity of the robot in local frame is

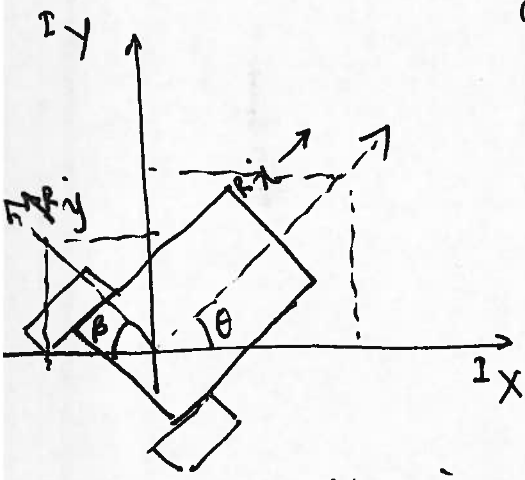
$$ {}^R\dot\theta = \frac{\dot\phi_r \cdot r}{2l} - \frac{\dot\phi_l \cdot r}{2l}.$$

$$
{}^{R}\dot{\xi}_{I} = \begin{bmatrix} \frac{r\cdot\dot{\varphi}_r}{2} + \frac{r\cdot\dot{\varphi}_l}{2} \\ 0 \\ \frac{\dot{\varphi}_r\cdot r}{2l} - \frac{\dot{\varphi}_l\cdot r}{2l} \end{bmatrix}
$$

Step 2: Map velocities from local frame to global frame.



Let ${}^{R}\dot{\xi}_{I} = \begin{bmatrix} {}^{R}\dot{x} \\ {}^{R}\dot{y} \\ {}^{R}\dot{\theta} \end{bmatrix}$ be the velocities in the robot local frame. We need to find their projections in the inertial frame.



Let the local y axis of the robot make an angle β w.r.t. inertial x-axis as shown in figure. From visual inspection we see that

$$\beta + \theta + 90^\circ = 180^\circ$$

$$\Rightarrow \beta = 90 - \theta$$

The velocity in the x-direction in the inertial axis is a sum of the projections of the velocities in the x and y direction in the robot local axis.

x-projection of ${}^{R}\dot{x}$ $= {}^{R}\dot{x}\cdot\cos\theta$.

x-projection of ${}^{R}\dot{y}$ $= -{}^{R}\dot{y}\cdot\cos\beta$ (negative because its in the negative ^{I}X direction)

$= -{}^{R}\dot{y}\cos(90-\theta)$

$= -{}^{R}\dot{y}\cdot\sin\theta$.

Therefore,

$$^{I}\dot{x} = {}^{R}\dot{x} \cdot \cos\theta - {}^{R}\dot{y} \cdot \sin\theta.$$

Similarly, $^{I}\dot{y} = {}^{R}\dot{x} \cdot \sin\theta + {}^{R}\dot{y} \cdot \cos\theta.$

From visual inspection,

$$^{I}\dot{\theta} = {}^{R}\dot{\theta}$$

Therefore,

$$^{I}\dot{\xi}_{s} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot {}^{R}\dot{\xi}_{s}$$

$$\boxed{^{I}\dot{\xi}_{s} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_{r}}{2} + \frac{r\cdot\dot{\phi}_{l}}{2} \\ 0 \\ \frac{r\cdot\dot{\phi}_{r}}{2l} - \frac{r\cdot\dot{\phi}_{l}}{2l} \end{bmatrix}}$$

NOTE: From the textbook, orthogonal rotation matrix $R(\theta)$ maps the motion along the robot's inertial frame to the robot's local frame. $^{R}\dot{\xi}_{s} = R(\theta).{}^{I}\dot{\xi}_{s}^{0}$. Therefore, we can derive the global frame robot motion by calculating the inverse of the rotation matrix. $^{I}\dot{\xi}_{s} = R^{-1}(\theta).{}^{R}\dot{\xi}_{s}$

For clarity, you can calculate $R(\theta)$ and $^{I}\dot{\xi}_{s}$ from there as a home exercise.

INVERSE KINEMATICS

Calculate the joint angles given the ~~position~~ velocities in the inertial frame.

From the forward kinematics,

$$^I\dot{\xi}_g = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{r\dot{\phi}_r}{2} + \frac{r\dot{\phi}_l}{2} \\ 0 \\ \frac{r\cdot\dot{\phi}_r}{2l} - \frac{r\cdot\dot{\phi}_l}{2l} \end{bmatrix} \quad \text{——} \quad (1)$$

Given $= \; ^I\dot{\xi}_g, r, l, \theta$

To compute $= \dot{\phi}_r, \dot{\phi}_l$

$$^I\dot{\xi}_g = \begin{bmatrix} ^I\dot{x} \\ ^I\dot{y} \\ ^I\dot{\theta} \end{bmatrix}.$$

From (1),

$$^I\dot{x} = \frac{r\cdot\dot{\phi}_r}{2}\cos\theta \;\cancel{+\frac{r\cdot\dot{\phi}_l}{2}\cdot\sin\theta} + \frac{r\cdot\dot{\phi}_l}{2}\cdot\cos\theta \quad \text{—} (2)$$

$$^I\dot{\theta} = \frac{r\cdot\dot{\phi}_r}{2l} - \frac{r\cdot\dot{\phi}_l}{2l}. \quad \text{——} \quad (3)$$

$$^I\dot{y} = \frac{r\cdot\dot{\phi}_r}{2}\cdot\sin\theta + \frac{r\cdot\dot{\phi}_l}{2}\cdot\sin\theta. \quad \text{——} \quad (4).$$

(2) $\Rightarrow$ $$^I\dot{x} = \frac{r\cdot\cos\theta}{2}(\dot{\phi}_r + \dot{\phi}_l)$$

$$\Rightarrow \dot{\phi}_r + \dot{\phi}_l = \frac{2\cdot{}^I\dot{x}}{r\cdot\cos\theta} \quad \text{——} \quad (5)$$

3 $\Rightarrow$ $${}^I\dot{\theta} = \frac{r}{2l}\left(\dot{\phi}_r - \dot{\phi}_l\right)$$

$$\Rightarrow \dot{\phi}_r - \dot{\phi}_l = \frac{{}^I\dot{\theta}}{r} \cdot 2l \qquad (6)$$

5 + 6 $\Rightarrow$ $$2 \cdot \dot{\phi}_r = \frac{2 \cdot {}^I\dot{x}}{r \cdot \cos\theta} + \frac{{}^I\dot{\theta} \cdot 2l}{r}$$

$$\dot{\phi}_r = \frac{{}^I\dot{x}}{r \cdot \cos\theta} + \frac{{}^I\dot{\theta}}{r}.$$

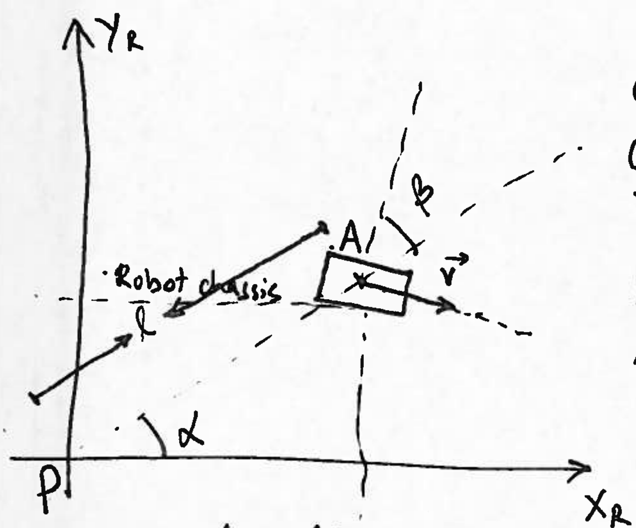
5 − 6 $\Rightarrow$ $$2 \cdot \dot{\phi}_l = \frac{2 \cdot {}^I\dot{x}}{2r \cdot \cos\theta} - \frac{{}^I\dot{\theta} \cdot 2l}{r}$$

$$\Rightarrow \dot{\phi}_l = \frac{{}^I\dot{x}}{r\cos\theta} - \frac{{}^I\dot{\theta} \cdot l}{r}$$

NOTE: We had three equations and two unknowns. You can use ${}^I\dot{y}$ to calculate $\dot{\phi}_r$ & $\dot{\phi}_l$ also. Try this as a home exercise.

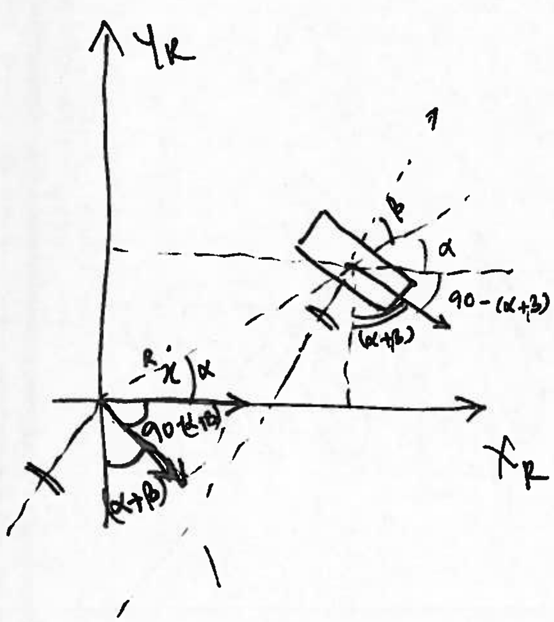
WHEEL KINEMATIC CONSTRAINTS

We made several simplifying assumptions in deriving the kinematics for ~~the too~~ a wheeled robot previously. In this section, we will relax the assumptions regarding the wheels and their alignment w.r.t. the body.



The fixed standard wheel has no vertical axis of rotation (no steering). Let P be the center of gravity of the robot. The wheel is at a distance 'l' from P in the robot frame. The angle of the wheel plane relative to the chassis is β. β is fixed for a fixed standard wheel. To compute the forward kinematics of the robot because of this wheel. Let the wheel spin at an angular velocity $\dot{\phi}$. Let the wheel radius be 'r'.

$$\vec{V} = r \cdot \dot{\phi}$$



x-component of ${}^R\dot{x}$ along the wheel is

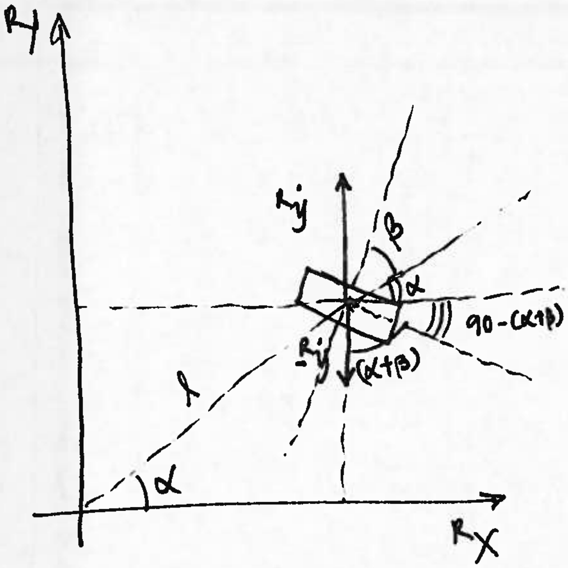
$$ {}^R\dot{x} \cdot \cos[90-(\alpha+\beta)]$$

$$= {}^R\dot{x} \sin(\alpha+\beta).$$

y-component of ${}^R\dot{x}$ along wheel coordinate frame is

$$ {}^R\dot{x} \sin[90-(\alpha+\beta)]$$

$$= {}^R\dot{x} \cos(\alpha+\beta).$$



Let us now look at the contribution of $^R\dot{y}$. ~~toward~~ The component along the X-axis in the wheel frame is

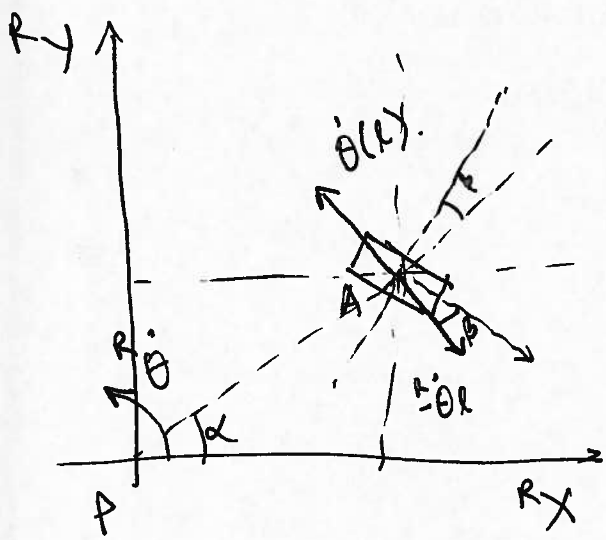
$$-{}^R\dot{y}\cdot\cos(\alpha+\beta).$$

And the contribution along the Y-axis of the wheel coordinate frame is

$$-{}^R\dot{y}\cdot(-1)\cdot\sin(\alpha+\beta)$$

$$= {}^R\dot{y}\,\sin(\alpha+\beta)$$

Let us now look at the contribution of the angular velocity



The linear velocity at point A due to angular velocity $^R\dot{\theta}$ is $^R\dot{\theta}\cdot l$ in the counter-clockwise direction i.e. it is $-{}^R\dot{\theta}\cdot l$ in the clockwise direction. Its component along the X-axis in the wheel frame is

$$-{}^R\dot{\theta}\cdot l\cdot\cos\beta.$$

Its contribution along the Y-axis in wheel frame is

$$(-1)\cdot -{}^R\dot{\theta}\cdot l\cdot\sin\beta.$$

$$= {}^R\dot{\theta}\cdot l\cdot\sin\beta.$$

We know that the wheel rotates at angular velocity $\dot{\phi}$ and has radius 'r'. Assuming no slippage, the sum of the contributions along the X-axis in wheel frame of ${}^R\dot{x}$, ${}^R\dot{y}$ & ${}^R\dot{\theta}$ should be equal to $r \cdot \dot{\phi}$

$$\Rightarrow \quad {}^R\dot{x} \sin(\alpha+\beta) - {}^R\dot{y} \cdot \cos(\alpha+\beta) - l \cdot {}^R\dot{\theta} \cdot \cos\beta = r\dot{\phi}$$

Similarly $\Rightarrow$ $$\begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & -l\cos\beta \end{bmatrix} {}^R\dot{\xi} = r\dot{\phi}$$

$$\Rightarrow \begin{bmatrix} \sin(\alpha+\beta) & -\cos(\alpha+\beta) & -l\cos\beta \end{bmatrix} R(\theta) \cdot {}^I\dot{\xi} = r\dot{\phi}$$

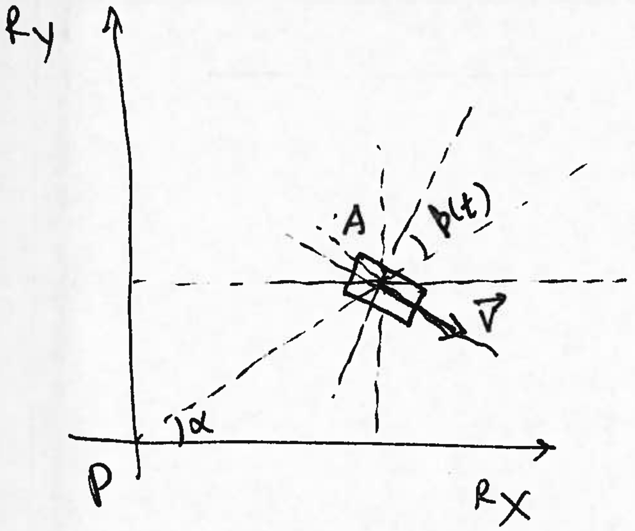
Similarly, we know that without slippage the velocity along Y-axis of wheel frame is zero.

$$\Rightarrow \quad {}^R\dot{x} \cos(\alpha+\beta) + {}^R\dot{y} \sin(\alpha+\beta) + l \cdot {}^R\dot{\theta} \cdot \sin\beta = 0$$

$$\Rightarrow \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \end{bmatrix} {}^R\dot{\xi} = 0$$

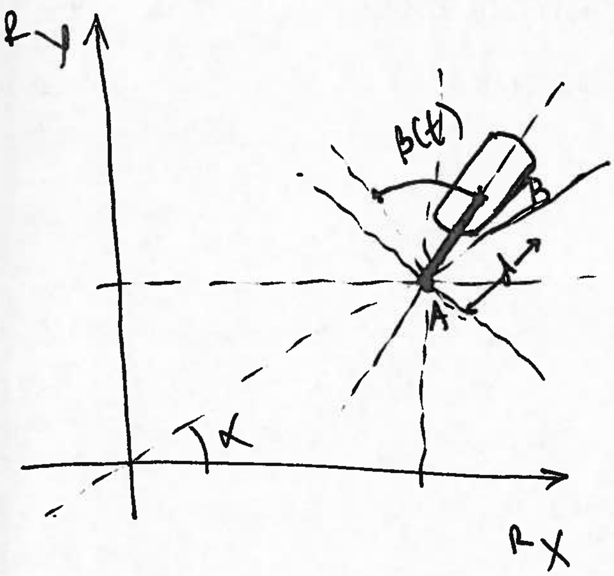
$$\Rightarrow \begin{bmatrix} \cos(\alpha+\beta) & \sin(\alpha+\beta) & l\sin\beta \end{bmatrix} \cdot R(\theta) \cdot {}^I\dot{\xi} = 0$$

STEERED STANDARD WHEEL



- Can rotate along vertical axis
- $\dot{\beta}$ does not affect instantaneous motion constraints of the robot [Homework exercise : ascertain this for yourself]

CASTOR WHEEL



- Rotates in the vertical axis at an offset 'd'.
- Two degrees of freedom $\dot{\phi}(t)$ and $\dot{\beta}(t)$

$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta\right] R(\theta)\dot{\xi}_I = r\dot{\varphi}$$

$$\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad \left(d + l\sin\beta\right)\right] R(\theta)\dot{\xi}_I + d\dot{\beta} = 0$$

- Motion orthogonal to wheel plane must be balanced by an equivalent and opposite amount of castor steering motion
- Steering action itself moves the robot chassis because of the offset between ground contact point & vertical axis of rotation