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2.4.1

$S = \{(X,Y) : X, Y \in \{1, \dots, 6\}\}$

$$\mathbb{P}(X+Y=10 \mid X+Y>8)$$

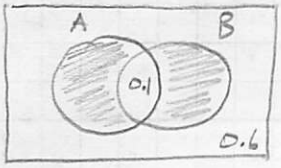
$$= \frac{\mathbb{P}(X+Y=10 \cap X+Y>8)}{\mathbb{P}(X+Y>8)}$$

$$= \frac{\mathbb{P}(X+Y=10)}{\mathbb{P}(X+Y>8)}$$

$$= \frac{\mathbb{P}(\{(4,6),(5,5),(6,4)\})}{\mathbb{P}\left(\left\{\begin{array}{l}(3,6),(4,5),(4,6),(5,4)\\(5,5),(5,6),(6,3),(6,4),(6,5)\\(6,6)\end{array}\right\}\right)}$$

$$= \boxed{3/10}$$

2.4.4 $\mathbb{P}(E \mid A\cup B) = \dfrac{\mathbb{P}(E\cap(A\cup B))}{\mathbb{P}(A\cup B)}$



$E = (A\cap B^c) \cup (A^c \cap B)$

$$\mathbb{P}(E \mid A\cup B) = \frac{\mathbb{P}(E)}{\mathbb{P}(A\cup B)} = \frac{0.3}{0.4}$$

$$= \boxed{3/4}$$

2.4.7 $D_i = R$: ith draw is red
$D_i = W$: ith draw is white

$$\mathbb{P}(D_1 = R \cap D_2 = R)$$

$$= \mathbb{P}(D_1=R)\,\mathbb{P}(D_2=R \mid D_1=R)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{3}{4}\right) = \boxed{\tfrac{3}{8}}$$

2.4.12

$\mathbb{P}$(At least 2 | At most 2)

$$= \frac{\mathbb{P}(\text{At least 2} \cap \text{At most 2})}{\mathbb{P}(\text{At most 2})}$$

$$= \frac{\mathbb{P}(\text{Exactly 2})}{\mathbb{P}(0, 1, \text{or } 2)} = \frac{3/8}{7/8} = \boxed{\tfrac{3}{7}}$$

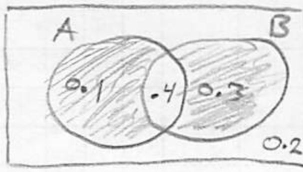
2.4.13 A = # on first die
B = # on second die

$A+B=8$ is the event: $\{(2,6), (3,5), (4,4), (5,3), (6,2)\}$

$$\mathbb{P}(A \geq 4 \mid A+B=8) = \boxed{\tfrac{3}{5}}$$

2.4.17

E = "At least one of A, B" $= A\cup B$
F = "At most one of A, B" $= (A\cap B)^c$



$$\mathbb{P}(E \mid F) = \mathbb{P}(A\cup B \mid (A\cap B)^c)$$

$$= \frac{\mathbb{P}((A\cup B)\cap(A\cap B)^c)}{\mathbb{P}[(A\cap B)^c]} = \frac{0.1+0.3}{0.1+0.3+0.2} = \frac{4}{6}$$

$$= \boxed{2/3}$$

2.4.22

$\mathbb{P}(3^{rd}$ key succeeds$) = \mathbb{P}(1^{st}$ fails $\cap\ 2^{nd}$ fails $\cap\ 3^{rd}$ succeeds$)$

$= \mathbb{P}(1^{st}$ fails$)\,\mathbb{P}(2^{nd}$ fails $\mid 1^{st}$ fails$)\,\mathbb{P}(3^{rd}$ succeeds $\mid 1^{st}$ & 2^{nd} failed$)$

$$= \left(\frac{n-1}{n}\right)\left(\frac{n-2}{n-1}\right)\left(\frac{1}{n-2}\right) = \boxed{\frac{1}{n}}$$

2.4.24 $D_i = W/B$: ith draw is white/black

$$\mathbb{P}(D_1=W, D_2=W, D_3=B, D_4=B, D_5=B)$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{3}{4}\right) = \boxed{\tfrac{1}{16}}$$

2.4.35 Experiment: toss coin, let other person call heads / tails.

$\mathbb{P}$(You win) $= \mathbb{P}$(You win with $\underbrace{\text{heads}}_{H}$ $\cup$ You win with $\underbrace{\text{tails}}_{T}$)

$= \mathbb{P}$(You win with H) $+ \mathbb{P}$(You win with T)
$= \mathbb{P}$(Call is T $\cap$ toss is H) $+ \mathbb{P}$(Call is H $\cap$ toss is T)
$= \mathbb{P}$(call is T) $\mathbb{P}$(toss is H | call is T)
$+ \mathbb{P}$(call is H) $\mathbb{P}$(toss is T | call is H)

$$= \tfrac{3}{10}\left(\tfrac{1}{2}\right) + \tfrac{7}{10}\left(\tfrac{1}{2}\right) = \boxed{\tfrac{1}{2}}$$

3.4.38 E = "early release", R = "related"

$$\mathbb{P}(E) = \mathbb{P}(E\cap R) + \mathbb{P}(E\cap R^c)$$
$$= \mathbb{P}(R)\mathbb{P}(E \mid R) + \mathbb{P}(R^c)\mathbb{P}(E \mid R^c)$$
$$= (.4)(.9) + (.6)(.01)$$
$$= .36 + .006 = \boxed{0.366}$$

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2.4.41

$$\mathbb{P}(\text{III} \mid R) = \frac{\mathbb{P}(\text{III} \cap R)}{\mathbb{P}(R)} = \frac{\mathbb{P}(\text{III})\,\mathbb{P}(R \mid \text{III})}{\mathbb{P}(\text{I})\mathbb{P}(R|\text{I}) + \mathbb{P}(\text{II})\mathbb{P}(R|\text{II}) + \mathbb{P}(\text{III})\mathbb{P}(R|\text{III})}$$

$$= \frac{(\frac{1}{3})(\frac{5}{8})}{(\frac{1}{3})(\frac{3}{8}) + (\frac{1}{3})(\frac{4}{8}) + (\frac{1}{3})(\frac{5}{8})} = \boxed{\frac{5}{12}}$$

2.4.42 F = "Red light flashes", L = "oil pressure is low"

$$\mathbb{P}(L \mid F) = \frac{\mathbb{P}(L \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(L \cap F)}{\mathbb{P}(F \cap L) + \mathbb{P}(F \cap L^c)} = \frac{\mathbb{P}(L)\mathbb{P}(F|L)}{\mathbb{P}(L)\mathbb{P}(F|L) + \mathbb{P}(L^c)\mathbb{P}(F|L^c)}$$

$$= \frac{(0.1)(0.99)}{(0.1)(0.99) + (0.9)(.02)} = \frac{0.099}{0.099 + 0.018} = \frac{99}{117} = \boxed{\frac{11}{13}}$$

2.4.46

F = "Basil ordered cherries flambé" $P(F) = 0.5$

M = "Basil ordered chocolate mousse" $P(M) = 0.4$

N = "Basil skips dessert" $P(N) = 0.1$

D = "Basil dies" $\mathbb{P}(D|F) = 0.6$, $\mathbb{P}(D|M) = 0.9$, $\mathbb{P}(D|N) = 0$

$$\mathbb{P}(F \mid D) = \frac{\mathbb{P}(F \cap D)}{\mathbb{P}(D)} = \frac{\mathbb{P}(F)\mathbb{P}(D|F)}{\mathbb{P}(F)\mathbb{P}(D|F) + \mathbb{P}(M)\mathbb{P}(D|M) + \cancel{\mathbb{P}(N)}\underbrace{\mathbb{P}(D|N)}_{0}}$$

$$= \frac{(0.5)(0.6)}{(0.5)(0.6) + (0.4)(0.9)} \approx 0.45$$

$$\mathbb{P}(M \mid D) = \ldots \approx 1 - 0.45 = 0.55$$

$\therefore$ $\boxed{\text{Margo is the prime suspect}}$

2.4.53 D_i = "Drawer i is selected", S = "First coin is silver"

$$\mathbb{P}(\text{"other coin is gold"} \mid S) = \mathbb{P}(D_3 \mid S)$$

$$= \frac{\mathbb{P}(D_3 \cap S)}{\mathbb{P}(S)} = \frac{\mathbb{P}(D_3)\,\mathbb{P}(S \mid D_3)}{\mathbb{P}(D_1)\mathbb{P}(S|D_1) + \mathbb{P}(D_2)\mathbb{P}(S|D_2) + \mathbb{P}(D_3)\mathbb{P}(S|D_3)}$$

$$= \frac{(1/3)(1/2)}{(\frac{1}{3})(0) + (\frac{1}{3})(1) + (\frac{1}{3})(\frac{1}{2})} = \frac{1/2}{1 + 1/2} = \boxed{\frac{1}{3}}$$

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2.5.2 C = "Spike passes chem." M = "Spike passes math."

$\mathbb{P}(C \cap M) = 0.12$

$\mathbb{P}(C)\mathbb{P}(M) = (0.35)(0.4) \neq 0.12$

C and M are not indep.

$\mathbb{P}(M^c \cap C^c) = \mathbb{P}((M \cup C)^c) = 1 - \mathbb{P}(M \cup C)$
$= 1 - [0.35 + 0.4 - 0.12] = \boxed{0.37}$

2.5.4 $\mathbb{P}(\text{same color})$
$= \mathbb{P}(\text{both red}) + \mathbb{P}(\text{both black}) + \mathbb{P}(\text{both white})$

$= \frac{3}{10}\cdot\frac{2}{9} + \frac{2}{10}\cdot\frac{4}{9} + \frac{5}{10}\cdot\frac{3}{9}$

$= \frac{6+8+15}{90} = \boxed{\frac{29}{90}}$

2.5.7 $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{8}$

(a) 1. $\mathbb{P}(A \cup B) = \frac{1}{4} + \frac{1}{8} = \boxed{\frac{3}{8}}$
2. $\mathbb{P}(A \cup B) = \frac{1}{4} + \frac{1}{8} - \frac{1}{4}\cdot\frac{1}{8} = \boxed{\frac{11}{32}}$

(b) 1. $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} = \boxed{0}$
2. $\mathbb{P}(A|B) = \mathbb{P}(A) = \boxed{\frac{1}{4}}$

2.5.14 $\mathbb{P}(A) = \frac{3}{6}$ $\mathbb{P}(B) = \frac{2}{6}$ $\mathbb{P}(C) = \frac{6}{36}$

$\mathbb{P}(A \cap B) = \mathbb{P}\left(\left\{\begin{matrix}(3,1) & (3,2)\\(4,1) & (4,2)\\(5,1) & (5,2)\end{matrix}\right\}\right) = \frac{6}{36} = \frac{1}{6}$

$\mathbb{P}(A \cap C) = \mathbb{P}\left(\left\{\begin{matrix}(3,4)\\(4,3)\\(5,2)\end{matrix}\right\}\right) = \frac{3}{36} = \frac{1}{12}$

$\mathbb{P}(B \cap C) = \mathbb{P}\left(\left\{\begin{matrix}(6,1)\\(5,2)\end{matrix}\right\}\right) = \frac{2}{36} = \frac{1}{18}$

$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{(5,2)\}) = \frac{1}{36}$

$\mathbb{P}(A)\mathbb{P}(B) = \frac{3}{6}\cdot\frac{2}{6} = \frac{1}{2}\cdot\frac{1}{3} = \frac{1}{6}$ ✓

$\mathbb{P}(A)\mathbb{P}(C) = \frac{3}{6}\cdot\frac{6}{36} = \frac{3}{36} = \frac{1}{12}$ ✓

$\mathbb{P}(B)\mathbb{P}(C) = \frac{2}{6}\cdot\frac{6}{36} = \frac{2}{36} = \frac{1}{18}$ ✓

$\mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C) = \frac{3}{6}\cdot\frac{2}{6}\cdot\frac{6}{36} = \frac{1}{36}$ ✓

Thus, A, B, and C are indep. events.

2.5.19
Let $p = \mathbb{P}(\text{win on any given try})$, $\mathbb{P}\binom{\text{not}}{\text{winning}} = 1-p$

$\mathbb{P}(\text{winning at least once in } 5)$
$= 1 - \mathbb{P}(\text{Not winning } \underline{\text{any}} \text{ of } 5)$
$= 1 - (1-p)^5$
$= 0.32$

$\Rightarrow (1-p)^5 = 0.68$
$1-p \approx 0.926$
$\Rightarrow \boxed{p \approx 0.074}$

2.5.20
$\mathbb{P}(A \text{ wins}) = \frac{1}{2} + \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right) + \cdots$

$= \frac{1}{2}\left(1 + \frac{1}{8} + \frac{1}{8^2} + \cdots\right) = \frac{1}{2}\cdot\frac{1}{1-1/8}$

$= \frac{1}{2}\cdot\frac{1}{7/8} = \frac{1}{2}\cdot\frac{8}{7} = \boxed{\frac{4}{7}}$

$\mathbb{P}(B \text{ wins}) = \frac{1}{4} + \left(\frac{1}{2}\right)^3\left(\frac{1}{4}\right) + \cdots = \frac{1}{4}\left(1 + \frac{1}{8} + \cdots\right)$

$= \frac{1}{4}\cdot\frac{1}{1-1/8} = \frac{1}{4}\cdot\frac{8}{7} = \boxed{\frac{2}{7}}$

$\mathbb{P}(C \text{ wins}) = \frac{1}{8} + \left(\frac{1}{2}\right)^3\left(\frac{1}{8}\right) + \cdots = \frac{1}{8}\left(1 + \frac{1}{8} + \cdots\right)$

$= \frac{1}{8}\cdot\frac{1}{1-1/8} = \frac{1}{8}\cdot\frac{8}{7} = \boxed{\frac{1}{7}}$

<u>Note</u>: $\frac{4}{7} + \frac{2}{7} + \frac{1}{7} = \frac{7}{7} = 1$

2.5.25
$\mathbb{P}(\text{At least one}) = 1 - \mathbb{P}(\text{none}) = 1 - \left(\frac{35}{36}\right)^n > \frac{1}{2}$

$\Rightarrow \left(\frac{35}{36}\right)^n < \frac{1}{2}$ $\Rightarrow n \ln\left(\frac{35}{36}\right) < \ln(1/2)$

negative

$\Rightarrow n > \frac{\ln(1/2)}{\ln(35/36)} \approx 24.6$

$\Longrightarrow \boxed{n = 25}$

2.5.26 $\mathbb{P}(\text{Sum}=7) = \frac{6}{36}$ $\mathbb{P}(\text{Sum}=8) = \frac{5}{36}$
$\mathbb{P}(\text{Neither 7 nor 8}) = 25/36$

$\mathbb{P}(8 \text{ before } 7) = \mathbb{P}(1^{st}=8) + \mathbb{P}(1^{st}=N,\ 2^{nd}=8)$
$+ \mathbb{P}(1^{st}=N,\ 2^{nd}=N,\ 3^{rd}=8)$
$+ \cdots$

$= \mathbb{P}(8) + \mathbb{P}(N)\mathbb{P}(8) + \mathbb{P}(N)\mathbb{P}(N)\mathbb{P}(8) + \cdots$

$= \frac{5}{36} + \frac{25}{36}\left(\frac{5}{36}\right) + \left(\frac{25}{36}\right)^2\left(\frac{5}{36}\right) + \cdots$

$= \frac{5}{36}\cdot\frac{1}{1-25/36} = \frac{5}{\cancel{36}}\cdot\frac{\cancel{36}}{11} = \boxed{\frac{5}{11}}$