Math 183 Bowers | Homework 3.2

3.2.4

Let X = # of the businesses audited

then $X \sim B(n,p)$, where

$n = 6$, $p = .153$

$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X \leq 1) = 1 - [\mathbb{P}(X=0) + \mathbb{P}(X=1)]$

$\mathbb{P}(X=0) = \binom{6}{0}(0.153)^0(0.847)^6$

$\mathbb{P}(X=1) = \binom{6}{1}(0.153)^1(0.847)^5$

$\boxed{\mathbb{P}(X \geq 2) = 1 - (0.847)^6 - 6(0.153)(0.847)^5}$

3.2.7

X = # of failing engines on two-engine plane

$X \sim B(n,p)$, where $n = 2$, $p = 0.4$

$\mathbb{P}(\text{lands safely}) = \mathbb{P}(X \leq 1) = 1 - \mathbb{P}(X = 2)$

$= 1 - \binom{2}{2}(0.4)^2(0.6)^0 = 0.84$

Y = # of failing engines on four-engine plane

$Y \sim B(n,p)$ with $n=4$, $p=0.4$

$\mathbb{P}(\text{lands safely}) = \mathbb{P}(Y \leq 2) = 1 - [\mathbb{P}(Y=3) + \mathbb{P}(Y=4)]$

$= 1 - \binom{4}{3}(0.4)^3(0.6)^1 - \binom{4}{4}(0.4)^4(0.6)^0$

≈ 0.82

$\boxed{\text{The slightly safer bet is the two-engine plane.}}$

3.2.9

Let X be the number of sixes rolled.

Case 1: 6 dice. $X \sim B(n,p)$; $n=6$, $p = 1/6$

$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0) = 1 - (5/6)^6 \approx 0.6651$

Case 2: 12 dice. $X \sim B(n,p)$; $n = 12$, $p = 1/6$

$\mathbb{P}(X \geq 2) = 1 - [\mathbb{P}(X=0) + \mathbb{P}(X=1)]$

$= 1 - (5/6)^{12} - \binom{12}{1}(1/6)(5/6)^{11} \approx 0.6187$

Case 3: 18 dice. $X \sim B(n,p)$; $n = 18$, $p = 1/6$

$\mathbb{P}(X \geq 3) = 1 - [\mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2)]$

$= 1 - \left[(5/6)^{18} + \binom{18}{1}(1/6)(5/6)^{17} + \binom{18}{2}(1/6)^2(5/6)^{16}\right]$

≈ 0.597

3.2.10

X = # of missiles that hit the plane.

$X \sim B(n,p)$ with $n = 6$, $p = .2$

$\mathbb{P}(X \geq 2) = 1 - \mathbb{P}(X \leq 1) = 1 - [\mathbb{P}(X=0) + \mathbb{P}(X=1)]$

$= 1 - \left[\binom{6}{0}(.2)^0(.8)^6 + \binom{6}{1}(.2)^1(.8)^5\right]$

≈ 0.3446

Y = # of rockets that hit the boat.

$Y \sim B(n,p)$ with $n = 10$, $p = 0.05$

$\mathbb{P}(Y \geq 1) = 1 - \mathbb{P}(Y=0) = 1 - (0.95)^{10}$

≈ 0.4012

The prob. the plane crashes is 0.34, while the prob. the boat sinks is 0.40.

$\boxed{\text{I would rather be on the plane.}}$

3.3.1

There are $\binom{5}{2} = 10$ total outcomes, all of which are equally likely.

(a) Let X = larger number drawn

k	1	2	3	4	5
$\mathbb{P}(X=k)$	0	$1/10$	$2/10$	$3/10$	$4/10$

(b) Let V = sum of the two numbers

k	3	4	5	6	7	8	9
$\mathbb{P}(V=k)$	$1/10$	$1/10$	$2/10$	$2/10$	$2/10$	$1/10$	$1/10$

3.3.2

There are $5^2 = 25$ equally likely outcomes

(a) X = larger number

k	1	2	3	4	5
$\mathbb{P}(X=k)$	$1/25$	$3/25$	$5/25$	$7/25$	$9/25$

(b) V = sum of the two numbers

k	2	3	4	5	6	7	8	9	10
$\mathbb{P}(V=k)$	$1/25$	$2/25$	$3/25$	$4/25$	$5/25$	$4/25$	$3/25$	$2/25$	$1/25$

3.3.5

A fair coin is tossed 3 times.

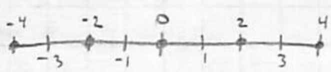
$X = (\text{\# of } Hs) - (\text{\# of } Ts)$

$S = \{ \underbrace{HHH}_{X=3}, \underbrace{HHT\ HTH\ THH}_{X=1}, \underbrace{HTT\ THT\ TTH}_{X=-1}, \underbrace{TTT}_{X=-3} \}$

X can be any number in the set $\{-3, -1, 1, 3\}$

k	-3	-1	1	3
$\mathbb{P}(X=k)$	$1/8$	$3/8$	$3/8$	$1/8$

3.3.7



Note that each step causes the particle to move from an even number to an odd, or from an odd to an even. So, each step changes the "parity." We start on even and move four times:

$$\underset{\uparrow \atop \text{start}}{\text{even}} \xrightarrow{①} \text{odd} \xrightarrow{②} \text{even} \xrightarrow{③} \text{odd} \xrightarrow{④} \underset{\uparrow \atop \text{end}}{\text{even}}$$

This means we end on an even number. Let X = final position of the particle. Then $X \in \{-4, -2, 0, 2, 4\}$

Think of a move to the right as adding a $(+1)$ to current position and moving to the left as adding a (-1).

(cont.) ↗

3.3.7 (cont.)

0 to 4 : $+1+1+1+1$ (choose all four to be +)

0 to 2 : $\pm1\pm1\pm1\pm1$ (choose 3 to be +)

0 to 0 : $\pm1\pm1\pm1\pm1$ (choose 2 to be +)

0 to -2: $\pm1\pm1\pm1\pm1$ (choose 1 to be +)

0 to -4: $-1-1-1-1$ (choose zero to be +)

$$\mathbb{P}(X=4) = \binom{4}{4}\left(\tfrac{1}{2}\right)^4\left(\tfrac{1}{2}\right)^0 = \frac{1}{16}$$
$$\mathbb{P}(X=2) = \binom{4}{3}\left(\tfrac{1}{2}\right)^3\left(\tfrac{1}{2}\right)^1 = 4/16$$
$$\mathbb{P}(X=0) = \binom{4}{2}\left(\tfrac{1}{2}\right)^2\left(\tfrac{1}{2}\right)^2 \quad 6/16$$
$$\mathbb{P}(X=-2) = \binom{4}{1}\left(\tfrac{1}{2}\right)^1\left(\tfrac{1}{2}\right)^3 = 4/16$$
$$\mathbb{P}(X=-4) = \binom{4}{0}\left(\tfrac{1}{2}\right)^0\left(\tfrac{1}{2}\right)^4 = 1/16$$

Formula: $\mathbb{P}(X=k) = \binom{4}{\frac{k}{2}+2}\left(\tfrac{1}{2}\right)^{\frac{k}{2}+2}\left(\tfrac{1}{2}\right)^{2-k/2}$ for $k \in \{-4, -2, 0, 2, 4\}$

OR: $\mathbb{P}(X = 2n-4) = \binom{4}{n}\left(\tfrac{1}{2}\right)^n\left(\tfrac{1}{2}\right)^{4-n}$ for $n \in \{0, 1, 2, 3, 4\}$

$[\ k = 2n-4\ ]$

3.3.8

Using the first formula:

$$\mathbb{P}(X=k) = \binom{4}{2+k/2}\left(\frac{2}{3}\right)^{2+k/2}\left(\frac{1}{3}\right)^{2-k/2}$$

3.3.11

$X \sim B(n,p)$ with $n=4$ and $p = 2/3$

So

$$\mathbb{P}(X=k) = \binom{4}{k}\left(\frac{2}{3}\right)^k\left(1/3\right)^{4-k}.$$

Let $Y = 2X+1$. Then

$$\mathbb{P}(Y=m) = \mathbb{P}(2X+1=m) = \mathbb{P}\left(X = \frac{m-1}{2}\right)$$
$$= \binom{4}{\frac{m-1}{2}}\left(2/3\right)^{\frac{m-1}{2}}\left(\frac{1}{3}\right)^{4-\left(\frac{m-1}{2}\right)}$$

3.3.14

Note: $F_X(k) = \mathbb{P}(X \le k) = \mathbb{P}(X \le k-1) + \mathbb{P}(X=k)$
$= F_X(k-1) + p_X(k)$

So: $p_X(k) = F_X(k) - F_X(k-1)$

$$p_X(0) = \mathbb{P}(X=0) = \mathbb{P}(X \le 0) = F_X(0) = \boxed{0}$$
$$p_X(1) = F_X(1) - F_X(0) = \boxed{2/42}$$
$$p_X(2) = F_X(2) - F_X(1) = 6/42 - 2/42 = \boxed{4/42}$$
$$p_X(3) = F_X(3) - F_X(2) = \tfrac{12}{42} - \tfrac{6}{42} = \boxed{6/42}$$
$$p_X(4) = F_X(4) - F_X(3) = \tfrac{20}{42} - \tfrac{12}{42} = \boxed{8/42}$$
$$p_X(5) = F_X(5) - F_X(4) = \tfrac{30}{42} - \tfrac{20}{42} = \boxed{10/42}$$
$$p_X(6) = 1 - F_X(5) = 1 - \tfrac{30}{42} = \boxed{12/42}$$

Math 183 Bowers | Homework 3.4

3.4.1

$f(y) = 4y^3, \; 0 \le y \le 1$

$\mathbb{P}(0 \le Y \le \tfrac{1}{2}) = \int_0^{1/2} 4y^3\,dy = y^4 \Big|_0^{1/2} = \boxed{\tfrac{1}{16}}$

3.4.2

$f_Y(y) = \frac{2}{3} + \frac{2}{3}y, \; 0 \le y \le 1$

$\mathbb{P}(\tfrac{3}{4} \le Y \le 1) = \int_{3/4}^1 \left(\tfrac{2}{3} + \tfrac{2}{3}y\right)dy = \tfrac{2}{3}y + \tfrac{1}{3}y^2 \Big|_{3/4}^1$

$= \left(\tfrac{2}{3} + \tfrac{1}{3}\right) - \left(\tfrac{2}{\cancel{3}} \cdot \tfrac{\cancel{3}}{4} + \tfrac{1}{\cancel{3}} \cdot \tfrac{\cancel{9}\,3}{16}\right) = 1 - \left(\tfrac{8}{16} + \tfrac{3}{16}\right)$

$= 1 - \left(\tfrac{11}{16}\right) = \boxed{5/16}$

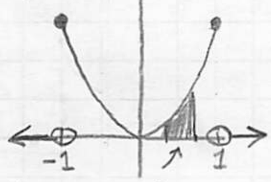
3.4.3

$f(y) = \frac{3}{2}y^2, \; -1 \le y \le 1$

$|y - \tfrac{1}{2}| < \tfrac{1}{4}$

$-\tfrac{1}{4} < Y - \tfrac{1}{2} < \tfrac{1}{4}$

$\tfrac{1}{4} < Y < 3/4$



$\mathbb{P}(|Y - \tfrac{1}{2}| < \tfrac{1}{4}) = \mathbb{P}(\tfrac{1}{4} < Y < 3/4)$

$= \int_{1/4}^{3/4} \tfrac{3}{2}y^2\,dy = \tfrac{1}{2}y^3 \Big|_{1/4}^{3/4} = \tfrac{1}{2}\left(\tfrac{3}{4}\right)^3 - \tfrac{1}{2}\left(\tfrac{1}{4}\right)^3$

$= \frac{27-1}{2(64)} = \frac{26}{2(64)} = \boxed{13/64}$

3.4.5

$f_Y(y) = 0.2e^{-0.2y}, \; y \ge 0$

(a) $\mathbb{P}(Y > 10) = \int_{10}^{\infty} 0.2e^{-0.2y}\,dy = -e^{-0.2y}\Big|_{10}^{\infty}$

$= \lim_{y\to\infty}(-e^{-0.2y}) - (-e^{-2}) = \boxed{e^{-2}}$

≈ 0.135

(b) $\mathbb{P}(X=1) = \binom{2}{1}(e^{-2})(1-e^{-2})$

≈ 0.234

3.4.8

$f_Y(y) = \lambda e^{-\lambda y}, \; y \ge 0$

If $y < 0$: $F_Y(y) = \mathbb{P}(Y \le y) = 0$

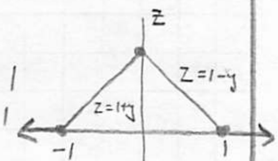
If $y \ge 0$:

$F_Y(y) = \int_0^y \lambda e^{-\lambda s}\,ds = -e^{-\lambda s}\Big|_0^y = 1 - e^{\lambda y}$

$$\boxed{F_Y(y) = \begin{cases} 1 - e^{\lambda y} & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases}}$$

3.4.9

$f(y) = \begin{cases} 0 & \text{if } |y| > 1 \\ 1 - |y| & \text{if } |y| < 1 \end{cases}$



$y < -1$: $F_Y(y) = 0$

$-1 \le y \le 0$: $F_Y(y) = \int_{-1}^y (1+s)\,ds = s + \frac{s^2}{2}\Big|_{-1}^y$

$= \left(y + \frac{y^2}{2}\right) - \left(-1 + \frac{1}{2}\right) = \frac{y^2}{2} + y + \frac{1}{2}$

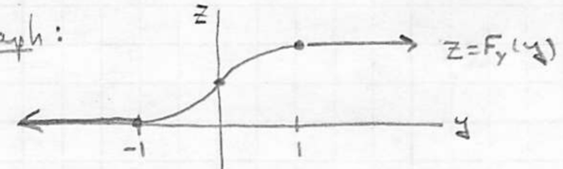
$0 \le y \le 1$: $F_Y(y) = \int_{-\infty}^y f(s)\,ds = \frac{1}{2} + \int_0^y (1-s)\,ds$

$= \frac{1}{2} + \left(s - \frac{s^2}{2}\right)\Big|_0^y = \frac{1}{2} + y - \frac{y^2}{2}$

$y > 1$: $F_Y(y) = 1$

$$\boxed{F_Y(y) = \begin{cases} 0 & \text{if } y < -1 \\ \frac{1}{2} + y + \frac{y^2}{2} & \text{if } -1 \le y \le 0 \\ \frac{1}{2} + y - \frac{y^2}{2} & \text{if } 0 \le y \le 1 \\ 1 & \text{if } y > 1 \end{cases}}$$

Graph:



3.4.11 $[e \approx 2.71]$

(a) $P(Y < 2) = F_Y(2) = \ln 2 \approx 0.693$

(b) $P(2 < Y \le 2.5) = F_Y(2.5) - F(2) = \ln(2.5) - \ln(2)$

(c) $P(2 < Y < 2.5) = \ln(2.5) - \ln 2 \approx 0.223$

(d) $f_Y(y) = \begin{cases} \frac{1}{y} & \text{if } 1 \le y \le e \\ 0 & \text{otherwise} \end{cases}$

3.4.14

$f_Y(y) = ye^{-y}, \; y \ge 0$

$F_Y(y) = \begin{cases} -ye^{-y} - e^{-y} & \text{if } y \ge 0 \\ 0 & \text{if } y < 0 \end{cases}$

Note: $\int ye^{-y}\,dy = -ye^{-y} + \int e^{-y}\,dy = -ye^{-y} - e^{-y}$ $\quad u = y \quad dv = e^{-y}dy \quad du = dy \quad v = -e^{-y}$

3.4.18

$f_Y(y) = \lambda e^{-\lambda y}, \; y \ge 0$

$F_Y(y) = 1 - e^{-\lambda y}, \; y \ge 0$

$h(y) = \frac{\lambda e^{-\lambda y}}{1 - [1 - e^{-\lambda y}]} = \frac{\lambda e^{-\lambda y}}{e^{-\lambda y}} = \boxed{\lambda}$ if $y \ge 0$