Math 183 | Homework 3.5

3.5.2

Black = odd numbers
Red = Even numbers
Green = zero, double zero.

Las Vegas:

$\mathbb{P}(\text{Red}) = \frac{18}{38}$, $\mathbb{P}(\text{Red}^c) = \frac{20}{38}$

$X_{LV} = \begin{cases} +1 & \text{if Red} \\ -1 & \text{otherwise} \end{cases}$

$E[X_{LV}] = (1)(\frac{18}{38}) + (-1)(\frac{20}{38}) = \frac{-2}{38} = \boxed{-\frac{1}{19}}$
≈ -0.05

Monte Carlo:

$\mathbb{P}(\text{Red}) = \frac{18}{37}$, $\mathbb{P}(\text{Red}^c) = \frac{19}{37}$

$X_{MC} = \begin{cases} +1 & \text{if Red} \\ -1 & \text{otherwise} \end{cases}$

$E[X_{MC}] = (+1)(\frac{18}{37}) + (-1)(\frac{19}{37}) = \boxed{-\frac{1}{37}} \approx -0.03$

We want B such that $B(E[X_{MC}] - E[X_{LV}]) > 3000$
OR

$B\left(-\frac{1}{37} + \frac{1}{19}\right) > 3000 \Rightarrow B > \boxed{117166.66}$

3.5.8

(a) $f_Y(y) = 3(1-y)^2$, $0 \le y \le 1$

$E[Y] = \int_0^1 y \cdot 3(1-y)^2 dy = \int_0^1 3y(1-2y+y^2)dy$
$= 3\int_0^1 (y - 2y^2 + y^3)dy = 3\left(\frac{y^2}{2} - \frac{2}{3}y^3 + \frac{y^4}{4}\right)\Big|_0^1$
$= 3\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = 3\left(\frac{6-8+3}{12}\right) = 3\left(\frac{1}{12}\right) = \boxed{\frac{1}{4}}$

(b) $f_Y(y) = 4ye^{-2y}$, $y \ge 0$

$E[Y] = \int_0^\infty 4y^2 e^{-2y} dy$ $\quad u = y^2 \quad dv = 4e^{-2y}dy$
$\quad du = 2y\,dy \quad v = -2e^{-2y}$

$= -2y^2e^{-2y} + \int 4ye^{-2y}dy \Big|_0^\infty$ $\quad u = y \quad dv = 4e^{-2y}dy$

$= -2y^2e^{-2y} + [-2ye^{-2y} + \int 2e^{-2y}dy]_0^\infty$ $\quad du = dy \quad v = -2e^{-2y}$

$= -2y^2e^{-2y} - 2ye^{-2y} - e^{-2y}\Big|_0^\infty = (0) - (-1) = \boxed{1}$

(c) $f_Y(y) = \begin{cases} 3/4 & 0 \le y \le 1 \\ 1/4 & 2 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$

$E[Y] = \int_0^1 \frac{3}{4}y\,dy + \int_2^3 \frac{1}{4}y\,dy = \frac{3}{8}y^2\Big|_0^1 + \frac{1}{8}y^2\Big|_2^3 = \frac{3}{8} + \frac{9-4}{8}$
$= \boxed{1}$

(d) $f_Y(y) = \sin(y)$, $0 \le y \le \pi/2$

$E[Y] = \int_0^{\pi/2} y \sin y\,dy$ $\quad u = y \quad dv = \sin y\,dy$
$\quad du = dy \quad v = -\cos y$

$= -y\cos y + \int \cos y\,dy \Big|_0^{\pi/2}$
$= -y\cos y + \sin y \Big|_0^{\pi/2}$
$= \left(\cancel{-\frac{\pi}{2}\cos\frac{\pi}{2}} + \sin\frac{\pi}{2}\right) - \cancel{(0 + \sin(0))} = \boxed{1}$

3.5.10 $f_Y(y) = \frac{1}{b-a}$, $a \le y \le b$

$E[Y] = \int_a^b y \frac{1}{b-a} dy = \frac{1}{b-a}\frac{y^2}{2}\Big|_a^b = \frac{b^2-a^2}{2(b-a)}$
$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{a+b}{2}}$

Since $E[Y]$ is the center of mass (or gravity) and since the mass is evenly distributed, it should be at the point midway between a and b, or $\frac{a+b}{2}$.

3.5.11 $f_Y(y) = \lambda e^{-\lambda y}$, $y > 0$

$E[Y] = \int_0^\infty y \cdot \lambda e^{-\lambda y} dy$ $\quad u = y \quad dv = \lambda e^{-\lambda y}dy$
$\quad du = dy \quad v = -e^{-\lambda y}$

$= -ye^{-\lambda y} + \int e^{-\lambda y}\Big|_0^\infty$
$= -ye^{-\lambda y} - \frac{e^{-\lambda y}}{\lambda}\Big|_0^\infty$
$= \lim_{y\to\infty}\underbrace{\left(-\frac{y}{e^{\lambda y}} - \frac{1}{\lambda e^{\lambda y}}\right)}_{\to 0} - \left(-0 - \frac{1}{\lambda}\right) = \boxed{\frac{1}{\lambda}}$

3.5.12 $f_Y(y) = 1/y^2$, $y \ge 1$

$E[Y] = \int_1^\infty y \cdot \frac{1}{y^2}dy = \int_1^\infty \frac{1}{y}dy = \ln y\Big|_1^\infty = \infty$

Note: $\int_{-\infty}^\infty f_Y(y)dy = \int_1^\infty \frac{1}{y^2}dy = -\frac{1}{y}\Big|_1^\infty = 1$

3.5.13

X = # of cars that pass the test
$X \sim B(n,p)$, where $n = 200$, $p = .8$

$E[X] = \sum_{k=0}^{200} k\mathbb{P}(X=k) = \sum_{k=0}^{\infty} k\binom{200}{k}(.8)^k(.2)^{200-k}$

$E[X] = np = (200)(.8) = 160.$

3.5.14

X = # of observations in $(\frac{1}{2}, 1)$
$X \sim B(n,p)$, where $n = 15$ and
$p = \mathbb{P}(\frac{1}{2} < Y < 1) = \int_{1/2}^1 3y^2dy = y^3\Big|_{1/2}^1 = 1 - \frac{1}{8} = \frac{7}{8}$

$E[X] = np = (15)(7/8) = \boxed{\frac{105}{8}}$

3.5.30

$f_Y(y) = 2(1-y)$, $0 \le y \le 1$

$E[Y^2] = \int_0^1 y^2 \cdot 2(1-y)dy = 2\int_0^1 (y^2 - y^3)dy$
$= 2\left(\frac{y^3}{3} - \frac{y^4}{4}\right)\Big|_0^1 = 2\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{2}{12} = \boxed{\frac{1}{6}}$

$f_W(w) = \left(\frac{1}{\sqrt{w}} - 1\right)$, $0 \le w \le 1$

$E[W] = \int_0^1 w \cdot \left(\frac{1}{\sqrt{w}} - 1\right)dw = \int_0^1 (w^{1/2} - w)dw$
$= \frac{2}{3}w^{3/2} - \frac{w^2}{2}\Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{4-3}{6} = \boxed{\frac{1}{6}}$

3.5.31

$f_Y(y) = 6e^{-6y}, \quad y > 0$

$Q(y) = 2(1 - e^{-2y})$

$$E[Q(Y)] = \int_0^\infty 2(1-e^{-2y}) \cdot 6e^{-6y}\,dy$$

$$= \int_0^\infty 12(e^{-6y} - e^{-8y})\,dy$$

$$= 12\left(-\frac{1}{6}e^{-6y} + \frac{1}{8}e^{-8y}\right)\Big|_0^\infty = 12\left(\frac{1}{6} - \frac{1}{8}\right)$$

$$= 12\left(\frac{8-6}{48}\right) = \frac{24}{48} = \boxed{1/2}$$

3.5.32

$f_Y(y) = 6y(1-y), \quad 0 < y < 1$

$V = 5Y^2$

$$E[V] = E[5Y^2] = \int_0^1 5y^2 \cdot 6y(1-y)\,dy$$

$$= \int_0^1 30(y^3 - y^4)\,dy = 30\left(\frac{y^4}{4} - \frac{y^5}{5}\right)\Big|_0^1$$

$$= 30\left(\frac{1}{4} - \frac{1}{5}\right) = 30\left(\frac{1}{20}\right) = \boxed{3/2}$$

3.6.2

$$f_Y(y) = \begin{cases} 3/4 & 0 \le y \le 1 \\ 1/4 & 2 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

$E[Y] = 1$ (See Exercise 3.5.8c)

$$E[Y^2] = \int_0^1 (y^2)\cdot\tfrac{3}{4}\,dy + \int_2^3 (y^2)\cdot\tfrac{1}{4}\,dy$$

$$= \tfrac{3}{12}y^3\Big|_0^1 + \tfrac{1}{12}y^3\Big|_2^3 = \tfrac{3}{12} + \tfrac{27-8}{12} = \tfrac{11}{6}$$

$$Var(Y) = 11/6 - (1)^2 = \tfrac{11-6}{6} = \boxed{5/6}$$

3.6.4 $X \sim U([0,1])$, $f_X(s) = \begin{cases} 1 & s \in [0,1] \\ 0 & \text{otherwise} \end{cases}$

$$E[X] = \tfrac{0+1}{2} = 1/2$$

$$E[X^2] = \int_0^1 s^2\,ds = \tfrac{1}{3}s^3\Big|_0^1 = 1/3$$

$$Var(X) = 1/3 - \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{3} - \tfrac{1}{4} = \boxed{\tfrac{1}{12}}$$

3.6.5

$f_Y(y) = 3(1-y)^2$, $0 < y < 1$

$E[Y] = \frac{1}{4}$ (See Exercise 3.5.8a)

$$E[Y^2] = \int_0^1 y^2\cdot 3(1-y)^2\,dy = \int_0^1 3y^2(1-2y+y^2)\,dy$$

$$= \int_0^1 3(y^2 - 2y^3 + y^4)\,dy = 3\left(\tfrac{y^3}{3} - \tfrac{1}{2}y^4 + \tfrac{y^5}{5}\right)\Big|_0^1$$

$$= 3\left(1/3 - \tfrac{1}{2} + \tfrac{1}{5}\right) = 3\left(\tfrac{10-15+6}{30}\right) = \tfrac{3}{30} = \tfrac{1}{10}$$

$$Var(Y) = \tfrac{1}{10} - \left(\tfrac{1}{4}\right)^2 = \tfrac{1}{10} - \tfrac{1}{16} = \tfrac{16-10}{160} = \tfrac{6}{160} = \boxed{\tfrac{3}{80}}$$

3.6.6

$f_Y(y) = \frac{2y}{K^2}$, $0 \le y \le K$

$$E[Y] = \int_0^K \tfrac{2y^2}{K^2}\,dy = \tfrac{2}{3K^2}y^3\Big|_0^K = \tfrac{2K^3}{3K^2} = \tfrac{2K}{3}$$

$$E[Y^2] = \int_0^K \tfrac{2y^3}{K^2}\,dy = \tfrac{y^4}{2K^2}\Big|_0^K = \tfrac{K^4}{2K^2} = \tfrac{K^2}{2}$$

$$Var(Y) = \tfrac{K^2}{2} - \left(\tfrac{2K}{3}\right)^2 = \tfrac{K^2}{2} - \tfrac{4K^2}{9} = \tfrac{9K^2 - 8K^2}{18} = \tfrac{K^2}{18}$$

$$Var(Y) = 2 \Rightarrow \tfrac{K^2}{18} = 2 \Rightarrow \boxed{K = 6} \quad (k \ge 0)$$

3.6.8 $f_Y(y) = 2y^{-3}$, $y \ge 1$

(a) $\int_1^\infty \frac{2}{y^3}\,dy = -\frac{1}{y^2}\Big|_1^\infty = 0 - (-1) = 1$

(b) $E[Y] = \int_1^\infty y\cdot\frac{2}{y^3}\,dy = \int_1^\infty \frac{2}{y^2}\,dy = -\frac{2}{y}\Big|_1^\infty = 2$

(c) $E[Y^2] = \int_1^\infty y^2\cdot\frac{2}{y^3}\,dy = \int_1^\infty \frac{2}{y}\,dy = \infty$

Thus, $Var(Y) = \infty$.

3.6.11 $f_Y(y) = \lambda e^{-\lambda y}$, $y \ge 0$

$E[Y] = 1/\lambda$ (See Exercise 3.5.11)

$$E[Y^2] = \int_0^\infty y^2\cdot\lambda e^{-\lambda y}\,dy \qquad u = y^2 \quad dv = \lambda e^{-\lambda y}dy$$

$$= -y^2 e^{-\lambda y} + \int 2y\, e^{-\lambda y}\,dy\Big|_0^\infty \qquad du = 2y\,dy \quad v = -e^{-\lambda y}$$

$$= -y^2 e^{-\lambda y} + \left(-\tfrac{2}{\lambda}\right)y e^{-\lambda y} + \int \tfrac{2}{\lambda} e^{-\lambda y}\,dy\Big|_0^\infty \qquad u = 2y \quad dv = e^{-\lambda y}dy$$

$$= -y^2 e^{-\lambda y} - \tfrac{2}{\lambda}y e^{-\lambda y} - \tfrac{2}{\lambda^2}e^{-\lambda y}\Big|_0^\infty \qquad du = 2dy \quad v = -\tfrac{1}{\lambda}e^{-\lambda y}$$

$$= (0) - \left(-2/\lambda^2\right) = 2/\lambda^2$$

$$Var(Y) = \tfrac{2}{\lambda^2} - \left(\tfrac{1}{\lambda}\right)^2 = \boxed{1/\lambda^2}$$

3.6.12 $Y \sim Exp(\lambda)$, $\lambda = 2$

$$P(Y > E(Y) + 2\sqrt{Var(Y)}) = P\left(Y > \tfrac{1}{\lambda} + 2\sqrt{1/\lambda^2}\right)$$

$$= P\left(Y > \tfrac{1}{\lambda} + 2\cdot\tfrac{1}{\lambda}\right) = P(Y > 3/\lambda) = P(Y > 3/2)$$

$$= \int_{3/2}^\infty f_Y(y)\,dy = \int_{3/2}^\infty 2e^{-2y}\,dy = -e^{-2y}\Big|_{3/2}^\infty = \boxed{e^{-3}}$$

3.6.16

We may use the following theorems:

$E[aX + b] = a\,E[X] + b$

$Var(aX + b) = a^2\,Var(X)$

Assume $E[W] = \mu$, $Var(W) = \sigma^2$

Then

$$E\left[\tfrac{W-\mu}{\sigma}\right] = E\left[\tfrac{1}{\sigma}W - \tfrac{\mu}{\sigma}\right] = \tfrac{1}{\sigma}E[W] - \tfrac{\mu}{\sigma} = 0$$

and

$$Var\left(\tfrac{W-\mu}{\sigma}\right) = Var\left(\tfrac{1}{\sigma}W - \tfrac{\mu}{\sigma}\right) = \left(\tfrac{1}{\sigma}\right)^2 Var(W) = 1.$$