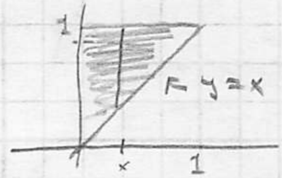
Math 183 | Homework 3.7 | Bowers

3.7.2 $f(x,y) = c(x^2+y^2)$ on unit square

$$\int_0^1\int_0^1 c(x^2+y^2)\,dx\,dy = \int_0^1 c\left(\frac{x^3}{3}+xy^2\right)\Big|_{x=0}^{x=1} dy = c\int_0^1\left(\frac{1}{3}+y^2\right)dy = c\left(\frac{1}{3}y+\frac{y^3}{3}\right)\Big|_0^1$$

$$= c\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{2}{3}c = 1 \quad\Rightarrow\quad \boxed{c = 3/2}$$

3.7.3 $f(x,y) = c(x+y)$ if $0<x<y<1$

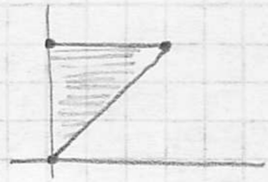


$$\int_0^1\int_x^1 c(x+y)\,dy\,dx = \int_0^1 c\left(xy+\frac{y^2}{2}\right)\Big|_x^1 dx$$

$$= \int_0^1 c\left(x+\frac{1}{2}-x^2-\frac{x^2}{2}\right)dx = \int_0^1 c\left(-\frac{3}{2}x^2+x+\frac{1}{2}\right)dx$$

$$= c\left(-\frac{1}{2}x^3+\frac{x^2}{2}+\frac{1}{2}x\right)\Big|_0^1 = c\left(-\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\right) = \frac{c}{2} = 1 \quad\Rightarrow\quad \boxed{c=2}$$

3.7.4 $f(x,y) = cxy$ over triangle



$$\int_0^1\int_x^1 cxy\,dy\,dx = \int_0^1 cx\frac{y^2}{2}\Big|_x^1 dx$$

$$= \int_0^1 \frac{c}{2}(x-x^3)\,dx = \frac{c}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_0^1$$

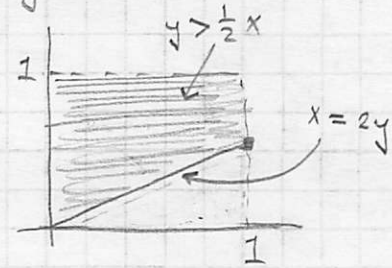
$$= \frac{c}{2}\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{c}{8} \quad\Rightarrow\quad \boxed{c=8}$$

3.7.5 $\mathbb{P}(X=x, Y=y) = \binom{3}{x}\binom{2}{y}\binom{4}{3-x-y}\Big/\binom{9}{3}$

X = # of white
Y = # of blue

$$x\in\{0,1,2,3\},\quad y\in\{0,1,2\},\quad 0\le x+y\le 3$$

3.7.13 $f(x,y) = x+y$ on unit square



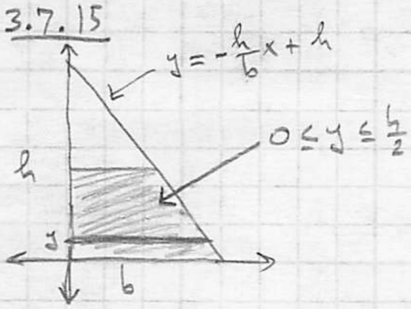
$$\mathbb{P}(X<2Y) = \mathbb{P}\left(Y>\frac{1}{2}X\right)$$

$$= \int_0^1\int_{x/2}^1 (x+y)\,dy\,dx = \int_0^1\left(xy+\frac{y^2}{2}\right)\Big|_{x/2}^1 dx$$

$$= \int_0^1\left(x+\frac{1}{2}-\frac{x^2}{2}-\frac{x^2}{8}\right)dx = \int_0^1\left(-\frac{5x^2}{8}+x+\frac{1}{2}\right)dx$$

$$= -\frac{5x^3}{24}+\frac{x^2}{2}+\frac{1}{2}x\Big|_0^1 = -\frac{5}{24}+\frac{1}{2}+\frac{1}{2} = \boxed{\frac{19}{24}}$$

3.7.15



Area of triangle $= \frac{bh}{2}$

$$f_{X,Y}(x,y) = \begin{cases}\frac{2}{bh} & \text{if } (x,y) \text{ is in triangle}\\ 0 & \text{otherwise}\end{cases}$$

$$P\left(Y\le\frac{h}{2}\right) = \int_0^{h/2}\int_0^{-\frac{b}{h}(y-h)}\frac{2}{bh}\,dx\,dy$$

$$= \int_0^{h/2}\frac{2}{bh}\left(\frac{b}{h}(h-y)\right)dy = \frac{2}{bh}\cdot\frac{b}{h}\left(hy-\frac{y^2}{2}\right)\Big|_0^{h/2}$$

$$= \frac{2}{h^2}\left(\frac{h^2}{2}-\frac{h^2}{8}\right) = 1-\frac{1}{4} = \boxed{\frac{3}{4}}$$

3.7.19(g) $f(x,y) = y e^{-xy-y} \quad x, y \ge 0$

$$f_X(x) = \int_0^\infty y e^{-y(x+1)}\,dy = -\left(\frac{1}{x+1}\right)^2 e^{-y(x+1)}\Big|_0^\infty = \boxed{\left(\frac{1}{x+1}\right)^2, \quad x \ge 0}$$

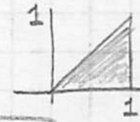
$$f_Y(y) = \int_0^\infty y e^{-xy} e^{-y}\,dx = y e^{-y}\int_0^\infty e^{-xy}\,dx = -e^{-y} e^{-x}\Big|_{x=0}^{x=\infty} = \boxed{e^{-y}, y \ge 0}$$

3.7.20 (a) $f(x,y) = \frac{1}{2}, \quad 0 \le x \le y \le 2$

$$f_X(x) = \int_x^2 \frac{1}{2}\,dy = \frac{1}{2}(2-x) = \boxed{1 - \frac{x}{2}, \quad 0 \le x \le 2}$$

$$f_Y(y) = \int_0^y \frac{1}{2}\,dx = \frac{1}{2}(y-0) = \boxed{y/2, \quad 0 \le y \le 2}$$

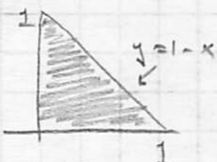
(b) $f(x,y) = \frac{1}{x}, \quad 0 \le y \le x \le 1$



$$f_X(x) = \int_0^x \frac{1}{x}\,dy = \frac{1}{x}(x-0) = \boxed{1, \quad 0 \le x \le 1}$$

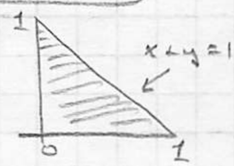
$$f_Y(y) = \int_y^1 \frac{1}{x}\,dx = \ln x\Big|_y^1 = -\ln y = \boxed{\ln\left(\frac{1}{y}\right), \quad 0 \le y \le 1}$$

(c) $f(x,y) = 6x, \quad 0 \le x \le 1, \quad 0 \le y \le 1-x$



$$f_X(x) = \int_0^{1-x} 6x\,dy = \boxed{6x(1-x), \quad 0 \le x \le 1}$$

$$f_Y(y) = \int_0^{1-y} 6x\,dx = 3x^2\Big|_0^{1-y} = \boxed{3(1-y)^2, \quad 0 \le y \le 1}$$



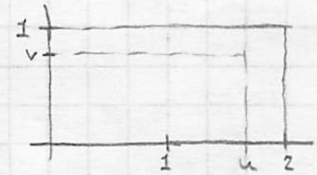
3.7.21 $f(x,y) = 6(1-x-y) \quad 0 \le x+y \le 1, \quad 0 \le x, y \le 1$

$$f_X(x) = \int_0^{1-x} 6(1-x-y)\,dy = 6\left(y - xy - \frac{y^2}{2}\right)\Big|_0^{1-x}$$

$$= 6\left((1-x) - x(1-x) - \frac{(1-x)^2}{2}\right) = 6\left((1-x)^2 - \frac{(1-x)^2}{2}\right) = \boxed{3(1-x)^2, \quad 0 \le x \le 1}$$

3.7.22 $f_Y(y) = \int_0^y 2e^{-x}e^{-y}\,dx = -2e^{-y}\cdot e^{-x}\Big|_0^y = -2e^{-y}(e^{-y}-1)$

$$= \boxed{2\left(e^{-y} - e^{-2y}\right), \quad y \ge 0}$$



3.7.27 (a) $f(x,y) = \frac{3}{2}y^2, \quad 0 \le x \le 2, \quad 0 \le y \le 1$

$$F(u,v) = \int_0^v \int_0^u \frac{3}{2}y^2\,dx\,dy = \int_0^v \left(\frac{3}{2}xy^2\Big|_{x=0}^{x=u}\right)dy$$

$$= \int_0^v \left(\frac{3}{2}uy^2\right)dy = \frac{1}{2}uy^3\Big|_0^v = \boxed{\frac{1}{2}uv^3, \quad 0 \le u \le 2, \quad 0 \le v \le 1}$$

(b) $f(x,y) = \frac{2}{3}(x+2y), \quad 0 \le x \le 1, \quad 0 \le y \le 1$

$$F(u,v) = \int_0^v\int_0^u \frac{2}{3}(x+2y)\,dx\,dy = \int_0^v \frac{2}{3}\left(\frac{x^2}{2} + 2xy\right)\Big|_0^u dy = \int_0^v \frac{2}{3}\left(\frac{u^2}{2} + 2uy\right)dy$$

$$= \frac{2}{3}\left(\frac{u^2}{2}y + uy^2\right)\Big|_0^v = \frac{2}{3}\left(\frac{u^2 v}{2} + uv^2\right) = \boxed{\frac{1}{3}\left(u^2 v + 2uv^2\right), \quad 0 \le u, v \le 1}$$

(c) $f(x,y) = 4xy, \quad 0 \le x, y \le 1$

$$F(u,v) = \int_0^v\int_0^u 4xy\,dx\,dy = 4\cdot\frac{x^2}{2}\Big|_0^u\cdot\frac{y^2}{2}\Big|_0^v = \boxed{u^2v^2, \quad 0 \le u, v \le 1}$$

Math 183 | Homework 3.7 | Bowers

3.7.31 $F(x,y) = k(4x^2y^2+5xy^4),\ 0<x,y<1 \qquad F(1,1)=k(9)=1 \Rightarrow k=\frac{1}{9}$

$$f(x,y) = \frac{\partial}{\partial y}\frac{\partial}{\partial x}F(x,y) = \frac{\partial}{\partial y}\frac{1}{9}(8xy^2+5y^4) = \boxed{\frac{1}{9}(16xy+20y^3),\ 0<x,y<1}$$

$$\mathbb{P}(0<X<\tfrac{1}{2},\ \tfrac{1}{2}<Y<1) = \int_{1/2}^{1}\int_0^{1/2}\frac{1}{9}(16xy+20y^3)\,dx\,dy$$

$$= \int_{1/2}^{1}\frac{1}{9}(8x^2y+20xy^3)\Big|_0^{1/2}dy = \int_{1/2}^{1}\frac{1}{9}(2y+10y^3)\,dy$$

$$= \frac{1}{9}\left(y^2+\frac{5}{2}y^4\right)\Big|_{1/2}^{1} = \frac{1}{9}\left(1+\frac{5}{2}-\frac{1}{4}-\frac{5}{32}\right) = \frac{1}{9}\left(\frac{99}{32}\right) = \boxed{\frac{11}{32}}$$

3.7.36 $f(x,y,z) = (x+y)e^{-z},\quad 0<x,y<1,\ z>0$

(a) $f_{X,Y}(x,y) = \int_0^\infty (x+y)e^{-z}\,dz = (x+y)\left(-e^{-z}\Big|_0^\infty\right) = \boxed{x+y,\quad 0<x,y<1}$

(b) $f_{Y,Z}(y,z) = \int_0^1 (x+y)e^{-z}\,dx = \left(\frac{x^2}{2}+xy\right)\Big|_0^1 e^{-z} = \boxed{\left(\frac{1}{2}+y\right)e^{-z},\quad 0<y<1,\ z>0}$

(c) $f_Z(z) = \int_0^1\left(\frac{1}{2}+y\right)e^{-z}\,dy = \left(\frac{1}{2}y+\frac{y^2}{2}\right)\Big|_0^1 e^{-z} = \boxed{e^{-z},\quad z>0}$

3.7.44

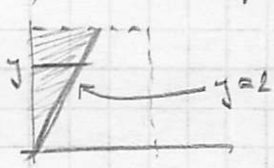
$$f_{X,Y}(x,y) = \frac{x}{2}\cdot 2y,\quad 0\le x\le 2,\ 0\le y\le 1$$

$$F(u,v) = \int_0^v\left(\int_0^u xy\,dx\right)dy = \left(\int_0^u x\,dx\right)\left(\int_0^v y\,dy\right)$$

$$= \left(\frac{x^2}{2}\Big|_0^u\right)\left(\frac{y^2}{2}\Big|_0^v\right) = \frac{u^2}{2}\cdot\frac{v^2}{2} = \boxed{\frac{1}{4}u^2v^2,\quad \begin{matrix}0\le u\le 2\\ 0\le v\le 1\end{matrix}}$$

3.7.45 $f_X(x) = 2x,\ 0\le x\le 1 \quad;\quad f_Y(y)=1,\ 0\le y\le 1 \quad;\quad X, Y$ indep

$$\mathbb{P}\left(\frac{Y}{X}>2\right) = \mathbb{P}(Y>2X) = \int_0^1\int_0^{y/2} 2x\,dx\,dy$$



$$= \int_0^1\left(x^2\Big|_0^{y/2}\right)dy = \int_0^1\frac{y^2}{4}\,dy$$

$$= \frac{y^3}{12}\Big|_0^1 = \boxed{\frac{1}{12}}$$

3.8.2 $f_X(x) = xe^{-x}$, $f_Y(y) = e^{-y}$, $x, y \geq 0$

$$f_{X+Y}(w) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{\neq 0 \text{ when } 0<x} \underbrace{f_Y(w-x)}_{\neq 0 \text{ when } x<w} dx = \int_0^w xe^{-x} \cdot e^{-(w-x)} dx$$

$$= \int_0^w xe^{-x} e^{-w} e^{x} dx$$

$$= e^{-w} \int_0^w x e^0 dx = e^{-w} \int_0^w x\,dx = \boxed{\frac{w^2}{2} e^{-w}, \quad w>0}$$

3.8.5 $Y \geq 0$ and $W = Y^2$

$$F_W(w) = \mathbb{P}(Y^2 \leq w) = \mathbb{P}(Y \leq \sqrt{w}) = \int_{-\infty}^{\sqrt{w}} f_Y(y)\,dy$$

$$f_W(w) = \underbrace{F_W'(w) = f_Y(\sqrt{w}) \cdot \frac{1}{2\sqrt{w}}}_{\text{Fundamental Theorem of Calculus}} = \boxed{\frac{1}{2\sqrt{w}} f_Y(\sqrt{w}), \quad w>0}$$

3.8.6 (Use # 3.8.5)

Y uniform over $[0,1]$ $\longrightarrow$ $f_Y(y) = \begin{cases} 1 & \text{if } 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$

$$f_W(w) = \frac{1}{2\sqrt{w}} f_Y(\sqrt{w}) = \begin{cases} \frac{1}{2\sqrt{w}} & \text{if } 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

3.8.9 (a) $f_X(x) = 1$, $f_Y(y) = 1$, $0 \leq x, y \leq 1$

$$f_{XY}(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X\left(\frac{w}{x}\right) \underbrace{f_Y(x)}_{=0 \text{ if } x \notin [0,1]} dx = \int_0^1 \frac{1}{x} f_X\left(\frac{w}{x}\right) \underbrace{f_Y(x)}_{=1} dx$$

$$= \int_0^1 \frac{1}{x} \underbrace{f_X\left(\frac{w}{x}\right)}\, dx = \int_w^1 \frac{1}{x}(1)\,dx = \ln x \Big|_w^1 = -\ln w$$

$$= \begin{cases} 1 & \text{if } 0 \leq \frac{w}{x} \leq 1 \Leftrightarrow 0 \leq w \leq x \\ 0 & \text{otherwise} \end{cases} \qquad = \boxed{\ln\left(\frac{1}{w}\right), \quad 0<w<1}$$

(Note: This is an indirect way to show $\underbrace{\int_0^1 \ln\left(\frac{1}{w}\right) dw}_{\text{Improper Integral}} = 1$)

(b) $f_X(x) = 2x$, $0 \leq x \leq 1$ and $f_Y(y) = 2y$, $0 \leq y \leq 1$

$$f_{XY}(w) = \int_{-\infty}^{\infty} \frac{1}{|x|} f_X\left(\frac{w}{x}\right) \underbrace{f_Y(x)}_{\Rightarrow 0 \leq x \leq 1} dx = \int_0^1 \frac{1}{x} f_X\left(\frac{w}{x}\right) \cdot 2x\,dx$$

$$= 2\int_0^1 \underbrace{f_X\left(\frac{w}{x}\right)}_{\Rightarrow 0 < \frac{w}{x} < 1 \;\Rightarrow\; 0<w<x} dx = 2\int_w^1 2\left(\frac{w}{x}\right) dx = 4w\int_w^1 \frac{1}{x}\,dx$$

$$= -4w \ln w = \boxed{4w \ln\left(\frac{1}{w}\right), \quad 0<w<1}$$