Math 183 Homework 3.9 Bowers

3.9.3 $f_{X,Y}(x,y) = \frac{2}{3}(x+2y)$, $0 \le x \le 1$, $0 \le y \le 1$

$$E[X+Y] = \int_0^1 \int_0^1 (x+y) \cdot \tfrac{2}{3}(x+2y)\,dx\,dy$$
$$= \tfrac{2}{3}\int_0^1\int_0^1 (x^2+3xy+2y^2)\,dx\,dy$$
$$= \tfrac{2}{3}\int_0^1 \left(\tfrac{x^3}{3} + \tfrac{3x^2y}{2} + 2xy^2\right)\Big|_{x=0}^{x=1} dy$$
$$= \tfrac{2}{3}\int_0^1 \left(\tfrac{1}{3} + \tfrac{3y}{2} + 2y^2\right) dy$$
$$= \tfrac{2}{3}\left(\tfrac{1}{3}y + \tfrac{3y^2}{4} + \tfrac{2y^3}{3}\right)\Big|_0^1 = \tfrac{2}{3}\left(\tfrac{1}{3}+\tfrac{3}{4}+\tfrac{2}{3}\right) = \boxed{\tfrac{7}{6}}$$

3.9.8 Let X be the number on the first die, and let Y be the number on the second die. Then X and Y are indep., so

$$E[XY] = E[X]E[Y] = (7/2)(7/2) = \boxed{\tfrac{49}{4}}$$

3.9.15 U is uniform on $[0,2\pi]$, so

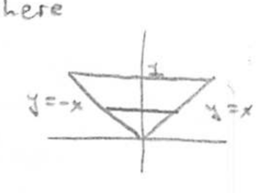
$$f_U(u) = \begin{cases} \frac{1}{2\pi} & \text{if } 0 \le u \le 2\pi, \\ 0 & \text{otherwise} \end{cases}$$

$X = \cos U$, $Y = \sin U$

$$E[XY] = \int_0^{2\pi} \tfrac{1}{2\pi}\cos u \sin u\,du = \tfrac{1}{2\pi}\int_0^{2\pi} \underbrace{\sin u}_{w} \cdot \underbrace{\cos u\,du}_{dw}$$
$$= \tfrac{1}{2\pi}\,\tfrac{(\sin u)^2}{2}\Big|_0^{2\pi} = 0$$
$$E[X] = \int_0^{2\pi} \tfrac{1}{2\pi}\cos u\,du = \tfrac{1}{2\pi}\sin u\Big|_0^{2\pi} = 0$$
$$E[Y] = \int_0^{2\pi} \tfrac{1}{2\pi}\sin u\,du = -\tfrac{1}{2\pi}\cos u\Big|_0^{2\pi} = -\tfrac{1}{2\pi}(1-1) = 0$$

Thus $\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y] = 0$, but X and Y are not indep., because $X^2+Y^2=1$.

3.9.16 $f_{X,Y}(x,y) = \begin{cases} 1 & -y<x<y,\ 0<y<1 \\ 0 & \text{elsewhere} \end{cases}$



$$E[XY] = \int_0^1 \left[\int_{-y}^{y} xy \cdot (1)\,dx\right] dy$$
$$= \int_0^1 \left[\tfrac{x^2y}{2}\Big|_{-y}^{y}\right] dy$$
$$= \int_0^1 \left(\tfrac{y^3}{2} - \tfrac{y^3}{2}\right) dy = \int_0^1 (0)\,dy = 0$$
$$E[X] = \int_0^1 \int_{-y}^{y} x\,dx\,dy = \int_0^1 \left(\tfrac{x^2}{2}\Big|_{-y}^{y}\right) dy = \int_0^1 \left(\tfrac{y^2}{2} - \tfrac{y^2}{2}\right) dy$$
$$= \int_0^1 (0)\,dy = 0$$

Thus, $\mathrm{Cov}(X,Y) = \underbrace{E[XY]}_{0} - \underbrace{E[X]}_{0}E[Y] = 0$.

Note: $E[Y] = \int_0^1\int_{-y}^{y} y\,dx\,dy = \int_0^1 xy\Big|_{-y}^{y} dy = \int_0^1 2y^2\,dy = \frac{2y^3}{3}\Big|_0^1 = \frac{2}{3}$ which is not zero.

3.9.16 (cont.) One way to show X and Y are not indep. is to compute the marginals:

$$f_X(x) = \begin{cases} \int_x^1 1\,dy = y\big|_x^1 = 1-x & \text{if } x \in [0,1] \\ \int_{-x}^1 1\,dy = y\big|_{-x}^1 = 1+x & \text{if } x \in [-1,0] \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} \int_{-y}^{y} 1\,dx = x\big|_{-y}^{y} = 2y & \text{if } y \in [0,1] \\ 0 & \text{otherwise.} \end{cases}$$

Since $f_{X,Y}(x,y) \ne f_X(x)f_Y(y)$, X and Y are not indep.

[You could also show $\mathbb{P}(X\in A, Y\in B) \ne \mathbb{P}(X\in A)\mathbb{P}(Y\in B)$ for some events A and B.]

3.9.18 $f_{X,Y}(x,y) = \frac{2}{3}(x+2y)$, $0 \le x \le 1$, $0 \le y \le 1$.

$$E[(X+Y)^2] = \int_0^1\int_0^1 (x+y)^2 \cdot \tfrac{2}{3}(x+2y)\,dx\,dy$$
$$= \tfrac{2}{3}\int_0^1\int_0^1 (x^2+2xy+y^2)(x+2y)\,dx\,dy$$
$$= \tfrac{2}{3}\int_0^1\int_0^1 (x^3 + 2x^2y + xy^2 + 2x^2y + 4xy^2 + 2y^3)\,dx\,dy$$
$$= \tfrac{2}{3}\int_0^1\int_0^1 (x^3+4x^2y+5xy^2+2y^3)\,dx\,dy$$
$$= \tfrac{2}{3}\int_0^1 \left(\tfrac{x^4}{4} + \tfrac{4x^3y}{3} + \tfrac{5x^2y^2}{2} + 2xy^3\right)\Big|_0^1 dy$$
$$= \tfrac{2}{3}\int_0^1 \left(\tfrac{1}{4} + \tfrac{4y}{3} + \tfrac{5y^2}{2} + 2y^3\right) dy = \tfrac{2}{3}\left(\tfrac{1}{4}y + \tfrac{2y^2}{3} + \tfrac{5y^3}{6} + \tfrac{y^4}{2}\right)\Big|_0^1$$
$$= \tfrac{2}{3}\left(\tfrac{1}{4}+\tfrac{2}{3}+\tfrac{5}{6}+\tfrac{1}{2}\right) = \tfrac{2}{3}\cdot\tfrac{3+8+10+6}{12} = \tfrac{2}{3}\cdot\tfrac{27}{12} = \tfrac{9}{6} = \tfrac{3}{2}$$

$E[X+Y] = \frac{7}{6}$ (from 3.9.3)

$$\mathrm{Var}(X+Y) = \tfrac{3}{2} - \left(\tfrac{7}{6}\right)^2 = \boxed{\tfrac{5}{36}}$$

Note: could also use $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\mathrm{Cov}(X,Y)$, which is

$$\left(E[X^2]-E[X]^2\right) + \left(E[Y^2]-E[Y]^2\right) + 2\left[E[XY]-E[X]E[Y]\right]$$

So... lots of computations.

3.9.20 $X \sim B(n, p_X)$, $Y \sim B(m, p_Y)$

Let $W = 4X + 6Y$. Then

$$E[W] = 4E[X] + 6E[Y] = 4np_X + 6mp_Y$$
$$\mathrm{Var}(W) = 4^2\mathrm{Var}(X) + 6^2\mathrm{Var}(Y) + 2(4)(6)\underbrace{\mathrm{cov}(X,Y)}_{0 \text{ (indep.)}}$$
$$= \boxed{4^2 np_X(1-p_X) + 6^2 mp_Y(1-p_Y)}$$

Note: $\mathrm{Cov}(X,Y) = 0$ because X and Y are indep. (which we are told in the problem)

4.3.1

(a) $F_Z(1.33) - F_Z(-0.44) = 0.9082 - 0.3300 = 0.5782$

(b) $F_Z(0.94) = 0.8264$

(c) $1 - F_Z(-1.48) = 1 - 0.0694 = 0.9306$

(d) $F_Z(-4.32) \approx 0$

4.3.2

(a) $F_Z(2.07) - F_Z(0) = 0.9808 - 0.5000 = 0.4808$

(b) $F_Z(-0.11) - F_Z(-0.64) = 0.4562 - 0.2611 = 0.1951$

(c) $1 - F_Z(-1.06) = 1 - 0.1446 = 0.8554$

(d) $F_Z(-2.33) = 0.0099$

(e) $1 - F_Z(4.61) \approx 1 - 1 = 0$

4.3.5

(a) -0.44

(b) 0.76

(c) 0.41

(d) 1.28

(e) 0.95

4.3.11

Let X be the number of deaths in SLC in the three months following their birthday. Then $X \sim B(n,p)$ with $n = 747$, $p = 3/12 = 1/4$

Note: We are assuming $p = 1/4$ because we want to see how likely our data are assuming that deaths are completely random throughout the year. Then a randomly chosen person would die in a 3 month span with prob. $3/12 = 1/4$.

So: Assuming $p = 1/4$, we would have 344 or more deaths in the 3-months after their birthday with prob. (continuity correction)

$$\mathbb{P}(X \geq 344) = \mathbb{P}(X \geq 343.5) = \mathbb{P}\left(\frac{X-\mu}{\sigma} \geq \frac{343.5 - 186.75}{\sqrt{140.06}}\right)$$

$\mu = np = 186.75$, $\sigma^2 = np(1-p) = 140.06$

$$= \mathbb{P}(Z \geq 13.25) \approx 0.$$

Since this prob. is so small, we might doubt that the deaths are completely random.

Note: This doubt does not suggest alternatives. It just means we might doubt our assumption that $p = 1/4$.

4.3.12

Let X be the number of correct answers. Assume answers are correct only by chance. Then $X \sim B(n,p)$ with $n = 1500$, $p = 1/5$

(There are 1500 guesses and 5 choices.)

Note: The number of test takers does not appear here, just the total number of guesses.

$\mu = np = 300$, $\sigma^2 = 240$, $\sigma \approx 15.49$

(cont.)

4.3.12 (cont.)

Assuming correct guesses are just chance ($p = 1/5$), how likely were the subjects to do at least as well as what we saw?

$$\mathbb{P}(X \geq 326) \underset{\uparrow}{=} \mathbb{P}(X \geq 325.5) = \mathbb{P}\left(Z \geq \frac{325.5 - 300}{15.49}\right)$$

(continuity correction)

$$= \mathbb{P}(Z \geq 1.65)$$
$$= 0.0495$$

This means, assuming answers are completely random, if we conduct this experiment many times, we expect our subjects to perform at least as well as they did about 4.95% of the time. This is quite unlikely, but not unbelievable.

4.3.15

Let $X_i = \begin{cases} 1 & \text{if } i\text{th toss is } H \\ 0 & \text{if } i\text{th toss is } T \end{cases}$ $\quad n = 200$, $p = 1/2$

$\mu = E[X_i] = p$, $\sigma^2 = Var(X_i) = p(1-p)$

Let $X = \sum_{i=1}^{200} X_i$. Then, by CLT,

$$X \approx N(n\mu, n\sigma^2) = N(100, 50)$$

So

$$\mathbb{P}(|X - E[X]| \leq 5) = \mathbb{P}(|X - n\mu| \leq 5)$$
$$= \mathbb{P}(-5 \leq X - n\mu \leq 5) = \mathbb{P}\left(\frac{-5}{\sqrt{50}} \leq \frac{X - n\mu}{\sqrt{n\sigma^2}} \leq \frac{5}{\sqrt{50}}\right)$$
$$= \mathbb{P}(-0.71 \leq Z \leq 0.71)$$
$$= F_Z(0.71) - F_Z(-0.71) = 0.7611 - 0.2420$$
$$= 0.5191$$

Also: $X \sim B(n,p)$ with $n = 200$, $p = 1/2$.

So $X \approx N(np, np(1-p)) = N(100, 50)$.

Thus, using the continuity correction,

$$\mathbb{P}(-5 \leq X - np \leq 5) = \mathbb{P}(-5.5 \leq X - np \leq 5.5)$$
$$= \mathbb{P}(-0.78 \leq Z \leq 0.78) = 0.5646$$

The difference is only the continuity correction.

4.3.16

Let $X_i = \#$ on ith die.

$\mu = E[X_i] = 7/2$, $\sigma^2 = 2.9 = (1.7)^2$

Let $X = \sum_{i=1}^{100} X_i$. Then $X \approx N(n\mu, n\sigma^2) = N(350, 17^2)$

So

$$\mathbb{P}(X \geq 371) \underset{\uparrow}{=} \mathbb{P}(X \geq 370.5)$$

(continuity correction)

$$= \mathbb{P}\left(\frac{X - n\mu}{\sqrt{n\sigma^2}} \geq \frac{370.5 - 350}{17}\right)$$

$$= \mathbb{P}(Z \geq 1.2) = 0.1151$$

Note: We used the continuity correction here because the problem explicitly said to use it.

4.3.17

Let X_i = the winnings on the i^{th} bet.

Then $X_i = \begin{cases} 5 & \text{if we win (lands on red)} \\ -5 & \text{if we lose} \end{cases}$

$\mu = E[X_i] = (5)(\frac{18}{38}) + (-5)(\frac{20}{38}) = -5/19$

$E[X_i^2] = (5)^2(\frac{18}{38}) + (-5)^2(\frac{20}{38}) = 25$

$\sigma^2 = 25 - (-5/19)^2 \approx 24.9$

Let $W = \sum_{i=1}^{100} X_i$ be the total winnings.

Then

$E[W] = 100\mu \approx -26.3$

$Var(W) = 100\sigma^2 \approx 2490 \approx (49.9)^2$

By CLT

$W \approx N(-26.3, (49.9)^2)$,

so

$\mathbb{P}(W \geq -50) = 1 - \mathbb{P}(W < -50)$
$= 1 - 0.3192 = \boxed{0.6808}$

4.3.23

Let X be the donations. Then

$\mathbb{P}(X > 30000) = \mathbb{P}(\frac{X-\mu}{\sigma} > \frac{30000 - 20000}{5000})$
$= \mathbb{P}(Z > 2.00) = \boxed{0.0228}$

4.3.24

Let Y be the number of days with child.

Then $Y \sim N(\mu, \sigma^2)$ with $\mu = 266$, $\sigma = 16$

Note: 10 months and 5 days $\approx$ 310 days

So:

$\mathbb{P}(Y \geq 310) = \mathbb{P}(\frac{Y-\mu}{\sigma} \geq \frac{310-266}{16})$
$= \mathbb{P}(Z \geq 2.75) = 0.0030$

Conclusion: It is very unlikely for a pregnancy to last 310 days or longer. It only happens about 0.3% of the time.

However: Considering how many babies are born every year, 0.3% might be a fairly large number!

Thus, I am not surprised that this reader had a pregnancy lasting 310 days.

4.3.26

Let A be the cross-sectional area.

Then $A \sim N(12.5, 0.2^2)$, so

$\underbrace{\mathbb{P}(12 \leq A \leq 13)}_{\text{Prob. tubing is okay}} = \mathbb{P}(\frac{12-12.5}{0.2} \leq \frac{A-\mu}{\sigma} \leq \frac{13-12.5}{0.2})$
$= \mathbb{P}(-2.5 \leq Z \leq 2.5)$
$= F_Z(2.5) - F_Z(-2.5)$
$= 0.9876$

NOW

Let X be the number of irregular tubes in a box of 1000. Then $X \sim B(n, p)$ with $n = 1000$ and

$p = \mathbb{P}(\text{irregular}) = 1 - \mathbb{P}(\text{okay})$
$= 0.0124$

Thus, the expected number of irregular tubes in the box is

$E[X] = np = \boxed{12.4}$

4.3.33

$Y_i \sim N(100, 16^2)$

Let $\bar{Y} = \frac{1}{9}\sum_{i=1}^{9} Y_i$. Then $\bar{Y} \sim N(100, \overset{Var(\bar{Y})}{\frac{16^2}{9}})$

Note: $Var(\bar{Y}) = \frac{16^2}{9}$, so $SD(\bar{Y}) = \frac{16}{3}$

So:

$\mathbb{P}(\bar{Y} > 103) = \mathbb{P}(\frac{\bar{Y}-100}{16/3} > \frac{103-100}{16/3})$
$= \mathbb{P}(Z > \frac{9}{16}) = \boxed{0.2877}$

Next: $\mathbb{P}(Y_i > 103) = \mathbb{P}(\frac{Y_i - 100}{16} > \frac{103-100}{16})$
$= \mathbb{P}(Z > \frac{3}{16}) = 0.4247$

Let X be the number of IQs that exceed 103. Then $X \sim B(n, p)$ with $n = 9$ and $p = 0.4247$.

So

$\mathbb{P}(X = 3) = \binom{9}{3}(0.4247)^3(0.5753)^6$
$= \boxed{0.23}$

Note: the prob. that an individual has an IQ greater than 103 is much more likely than the average IQ of a group exceeding 103. That is why

$\mathbb{P}(Y_i > 103) = 0.4247$ and
$\mathbb{P}(\bar{Y} > 103) = 0.2877$.