5.2.1 $p_X(k;\theta) = \theta^{k}(1-\theta)^{1-k}$; $k=0,1$, $0<\theta<1$

$L(\theta) = \prod_{i=1}^{k} \theta^{k_i}(1-\theta)^{1-k_i} = \theta^{\Sigma k_i}(1-\theta)^{n-\Sigma k_i}$

$\ln L(\theta) = (\Sigma k_i)\ln\theta + (n-\Sigma k_i)\ln(1-\theta)$

$\frac{d}{d\theta}\ln L(\theta) = \frac{\Sigma k_i}{\theta} - \frac{n-\Sigma k_i}{1-\theta} = 0$

$\frac{\Sigma k_i}{\theta} = \frac{n-\Sigma k_i}{1-\theta} \Rightarrow \theta = \frac{1}{n}\Sigma k_i$

Thus: $\boxed{\hat{\theta} = \frac{1}{n}\Sigma X_i = \bar{X},\ \theta_e = 5/8}$

5.2.3 $f_Y(y;\lambda) = \lambda e^{-\lambda y}$, $y \ge 0$

$L(\lambda) = \prod_{i=1}^{n} \lambda e^{-\lambda y_i} = \lambda^n e^{-\lambda \Sigma y_i}$

$\ln L(\lambda) = n\ln\lambda - \lambda\Sigma y_i$

$\frac{d}{d\lambda}\ln L(\lambda) = \frac{n}{\lambda} - \Sigma y_i = 0 \Rightarrow \lambda = \frac{n}{\Sigma y_i}$

$\boxed{\hat{\lambda} = \frac{n}{\Sigma Y_i} = \frac{1}{\bar{Y}},\ \lambda_e = \frac{4}{32.8} \approx 0.122}$

5.2.4 $p_X(k;\theta) = \frac{\theta^{2k}e^{-\theta^2}}{k!}$, $k=0,1,2,\ldots$

$L(\theta) = \prod_{i=1}^{n}\frac{\theta^{2k_i}e^{-\theta^2}}{k_i!} = \frac{\theta^{2\Sigma k_i}e^{-n\theta^2}}{\pi(k_i!)}$

$\ln L(\theta) = 2(\Sigma k_i)\ln\theta - n\theta^2 - \Sigma\ln(k_i!)$

$\frac{d}{d\theta}\ln L(\theta) = \frac{2\Sigma k_i}{\theta} - 2n\theta = 0 \Rightarrow \theta^2 = \frac{\Sigma k_i}{n}$

$\boxed{\hat{\theta} = \sqrt{\frac{\Sigma X_i}{n}} = \sqrt{\bar{X}}}$

5.2.5 $f_Y(y;\theta) = \frac{y^3 e^{-y/\theta}}{6\theta^4}$, $y \ge 0$

$L(\theta) = \prod_{i=1}^{n}\frac{y_i^3 e^{-y_i/\theta}}{6\theta^4} = \frac{(\prod y_i^3)\, e^{-\Sigma y_i/\theta}}{6^n\theta^{4n}}$

$\ln L(\theta) = \Sigma 3\ln y_i - \frac{1}{\theta}\Sigma y_i - \ln(6^n) - 4n\ln\theta$

$\frac{d}{d\theta}L(\theta) = \frac{1}{\theta^2}(\Sigma y_i) - \frac{4n}{\theta} = 0$

$\frac{\Sigma y_i}{\theta^2} = \frac{4n}{\theta} \Rightarrow \theta(\Sigma y_i) = 4n\theta^2$

$\Rightarrow 4n\theta^2 - (\Sigma y_i)\theta = 0$

$\Rightarrow \theta = 0$ or $\theta = \frac{\Sigma y_i}{4n}$

$\boxed{\text{Since } \theta \ne 0, \text{ we have } \hat{\theta} = \frac{\Sigma Y_i}{4n} = \frac{1}{4}\bar{Y}, \text{ and so } \theta_e = \frac{8.8}{12} \approx 0.73}$

5.2.6 $f_Y(y;\theta) = \frac{\theta}{2\sqrt{y}}e^{-\theta\sqrt{y}}$, $y \ge 0$

$L(\theta) = \prod_{i=1}^{n}\frac{\theta}{2\sqrt{y_i}}e^{-\theta\sqrt{y_i}} = \frac{\theta^n}{2^n}\frac{e^{-\theta\Sigma\sqrt{y_i}}}{\prod\sqrt{y_i}}$

$\ln L(\theta) = n\ln\theta - n\ln 2 - \theta\Sigma\sqrt{y_i} - \Sigma\frac{1}{2}\ln y_i$

$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta} - \Sigma\sqrt{y_i} = 0 \Rightarrow \theta = \frac{n}{\Sigma\sqrt{y_i}}$

$\boxed{\hat{\theta} = \frac{n}{\Sigma\sqrt{Y_i}},\ \theta_e \approx 0.453}$

5.2.7 $f_Y(y;\theta) = \theta y^{\theta-1}$, $0 \le y \le 1$, $\theta > 0$.

$L(\theta) = \prod\theta y_i^{\theta-1} = \theta^n(\prod y_i)^{\theta-1}$

$\ln L(\theta) = n\ln\theta + (\theta-1)\Sigma\ln y_i$

$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta} + \Sigma\ln y_i = 0 \Rightarrow \hat{\theta} = -\frac{n}{\Sigma\ln Y_i}$

$\theta_e = \frac{-5}{\ln .77 + \ln .82 + \ln .92 + \ln .94 + \ln .98} \approx \boxed{7.996}$

5.2.12 $f_Y(y;\theta) = \frac{2y}{\theta^2}$, $0 \le y \le \theta$

$L(\theta) = \prod\frac{2y_i}{\theta^2} = \frac{2^n}{\theta^{2n}}\prod y_i$

$\ln L(\theta) = n\ln 2 - 2n\ln\theta + \Sigma\ln y_i$

$\frac{d}{d\theta}\ln L(\theta) = -\frac{2n}{\theta} < 0$, so $L(\theta)$ is a decreasing function.

We want to minimize θ subject to the constraint $\theta \ge y$, so the smallest θ could be is the largest value of y. Thus

$\boxed{\hat{\theta} = Y_{MAX},\ \theta_e = 0.92}$

Out of order

5.2.11 $f_Y(y;\theta) = \frac{2y}{1-\theta^2}$, $\theta \le y \le 1$

$L(\theta) = \prod\frac{2y_i}{1-\theta^2} = \left(\frac{2}{1-\theta^2}\right)^n\prod y_i$

$\ln L(\theta) = n\ln 2 - n\ln(1-\theta^2) + \Sigma\ln y_i$

$\frac{d}{d\theta}\ln L(\theta) = \frac{-n}{1-\theta^2}(-2\theta) = \frac{2n\theta}{1-\theta^2}$

there are two critical points: $\theta = 0$, $\theta = 1$.

$\theta = 1$ is not in the domain of $L(\theta)$

$\theta = 0$ is a local minimum:

$\frac{d^2}{d\theta^2}\ln L(\theta)\Big|_{\theta=0} = \frac{2n(1-\theta^2) - 2n\theta(-2\theta)}{(1-\theta^2)}\Big|_{\theta=0}$

$= \frac{2n + 2n\theta^2}{1-\theta^2}\Big|_{\theta=0} = 2n > 0$

We want to maximize $L(\theta)$, so we minimize $1-\theta^2$ subject to the constraint $\theta \le y \le 1$.

We make θ as close to 1 as possible.

The largest θ can be is the smallest y value, so $\boxed{\hat{\theta} = Y_{min},\ \theta_e = 0.21}$

5.2.13 $f_Y(y;\theta) = \theta k^\theta\left(\frac{1}{y}\right)^{\theta+1}$; $y \ge k$, $\theta \ge 1$

$L(\theta) = \prod\theta k^\theta\left(\frac{1}{y^{\theta+1}}\right) = \theta^n k^{n\theta}(\prod y_i)^{-(\theta+1)}$

$\ln L(\theta) = n\ln\theta + n\theta\ln k - (\theta+1)\Sigma\ln y_i$

$\frac{d}{d\theta}\ln L(\theta) = \frac{n}{\theta} + n\ln k - \Sigma\ln y_i = 0$

$\boxed{\hat{\theta} = \frac{n}{(\Sigma\ln Y_i) - n\ln k}}$

If $n = 25$, then $\boxed{\hat{\theta} = \frac{25}{\left(\sum_{i=1}^{25}\ln Y_i\right) - 25\ln k}}$

5.2.16 $f_Y(y;\theta) = \frac{2y}{\theta^2};\ 0 \le y \le \theta$

$$E[Y] = \int_0^\theta y \cdot \frac{2y}{\theta^2}\,dy = \int_0^\theta \frac{2y^2}{\theta^2}\,dy$$

$$= \frac{2y^3}{3\theta^2}\Big|_0^\theta = \frac{2\theta^3}{3\theta^2} = \frac{2\theta}{3}$$

$$E[Y] = \bar{Y} \Rightarrow \frac{2\theta}{3} = \bar{Y} \Rightarrow \theta = \frac{3}{2}\bar{Y}$$

$$\boxed{\hat{\theta} = \frac{3}{2}\bar{Y}, \quad \theta_e = 75}$$

5.2.17 $f_Y(y;\theta) = (\theta^2+\theta)y^{\theta-1}(1-y),$
$0 \le y \le 1$

$$E[Y] = \int_0^1 y \cdot (\theta^2+\theta)y^{\theta-1}(1-y)\,dy$$

$$= (\theta^2+\theta)\int_0^1 y^\theta(1-y)\,dy$$

$$= (\theta^2+\theta)\int_0^1 \left[y^\theta - y^{\theta+1}\right]dy$$

$$= (\theta^2+\theta)\left[\frac{y^{\theta+1}}{\theta+1} - \frac{y^{\theta+2}}{\theta+2}\right]\Big|_0^1$$

$$= (\theta^2+\theta)\left[\frac{1}{\theta+1} - \frac{1}{\theta+2}\right] = (\theta^2+\theta)\cdot\frac{1}{(\theta+1)(\theta+2)}$$

$$= \frac{\theta}{\theta+2}$$

$$E[Y] = \bar{Y} \Rightarrow \frac{\theta}{\theta+2} = \bar{Y} \Rightarrow \theta = \bar{Y}\theta + 2\bar{Y}$$
$$\Rightarrow \theta(1-\bar{Y}) = 2\bar{Y}$$
$$\Rightarrow \theta = 2\bar{Y}/(1-\bar{Y})$$

$$\boxed{\hat{\theta} = \frac{2\bar{Y}}{1-\bar{Y}}, \quad \theta_e = \frac{2\cdot\frac{1}{n}\Sigma y_i}{1-\frac{1}{n}\Sigma y_i}}$$

5.2.21 $f_Y(y;\theta) = \theta k^\theta y^{-(\theta+1)},\ y \ge k,\ \theta \ge 1$

$$E[Y] = \int_k^\infty y \cdot \theta k^\theta y^{-(\theta+1)}\,d\theta$$

$$= \theta k^\theta \int_k^\infty y^{-\theta}\,d\theta$$

$$= \theta k^\theta \frac{y^{-\theta+1}}{-\theta+1}\Big|_k^\infty = \theta k^\theta\left[\cancel{\lim_{y\to\infty}\frac{y^{-\theta+1}}{-\theta+1}}^{\,0} - \frac{k^{-\theta+1}}{-\theta+1}\right]$$

$$= \theta k^\theta\left[\frac{k^{-\theta+1}}{\theta-1}\right] = \frac{\theta k}{\theta-1}$$

$$E[Y] = \bar{Y} \Rightarrow \frac{\theta k}{\theta-1} = \bar{Y} \Rightarrow \theta k = \theta\bar{Y} - \bar{Y}$$

$$\Rightarrow \theta(k-\bar{Y}) = -\bar{Y} \Rightarrow \theta = -\frac{\bar{Y}}{k-\bar{Y}}$$

$$\boxed{\hat{\theta} = \frac{\bar{Y}}{\bar{Y}-k}, \quad \theta_e = \frac{\bar{y}}{\bar{y}-k} \quad \text{where } \bar{y} = \frac{1}{n}\Sigma y_i}$$