Math 183 | Homework 5.3 | Bowers

5.3.1

$\bar{y} = 107.9$, $\sigma = 15$, $n = 50$

$ME = z^* \frac{\sigma}{\sqrt{n}} = (1.96)\cdot\frac{15}{\sqrt{50}} \approx 4.16$

The CI is

$(\bar{y} - ME, \bar{y} + ME) = (107.9 - 4.16, 107.9 + 4.16)$
$= (103.74, 112.06)$

5.3.3

$\bar{y} = 70.83$, $\sigma = 8$, $n = 6$

$ME = z^* \frac{\sigma}{\sqrt{n}} = (1.96)\cdot\frac{8}{\sqrt{6}} \approx 6.40$

CI: $(70.83 - 6.40, 70.83 + 6.40) = (64.43, 77.23)$

Since 80 days is not in the CI, we conclude that the average CH_3^{203} half-life is probably not the same for men and women.

5.3.4

$\bar{y} = 188.4$, $\sigma = 40.7$, $n = 38$

$ME = z^* \frac{\sigma}{\sqrt{n}} = (1.96)\cdot\frac{40.7}{\sqrt{38}} \approx 12.9$

CI: $(188.4 - 12.9, 188.4 + 12.9) = (175.5, 201.3)$

Since 192.0 is in the CI, there is no reason to assume the diet has any effect.

5.3.7

Let $p = \mathbb{P}(\bar{Y} - 0.96\cdot\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{Y} + 1.06\cdot\frac{\sigma}{\sqrt{n}})$

$= \mathbb{P}(-0.96\frac{\sigma}{\sqrt{n}} \le \mu - \bar{Y} \le 1.06\frac{\sigma}{\sqrt{n}})$

$= \mathbb{P}(-0.96 \le \frac{\mu - \bar{Y}}{\sigma/\sqrt{n}} \le 1.06)$

$= \mathbb{P}(-1.06 \le \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \le 0.96)$

Since $Y_i \sim N(\mu, \sigma^2)$, we have $\frac{\bar{Y} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

Thus $Z = \frac{\bar{Y} - \mu}{\sigma/\sqrt{n}}$, so

$p = \mathbb{P}(-1.06 \le Z \le 0.96) = 0.6869$

Let X be the number of intervals that contain μ. Then $X \sim B(n,p)$ with $n = 5$ and $p = 0.6869$. So

$\mathbb{P}(X \ge 4) = \mathbb{P}(X=4) + \mathbb{P}(X=5)$

$= \binom{5}{4}(0.6869)^4(0.3131) + \binom{5}{5}(0.6869)^5(0.3131)^0$

$= 5(0.6869)^4(0.3131) + (0.6869)$

≈ 0.501

5.3.19

$\hat{p} = 0.59$, $n = 998$

$ME = z^* SE(\hat{p}) = z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = (1.96)\sqrt{\frac{(0.59)(.41)}{998}} \approx 0.03$

Thus, the margin of error is $\approx 3\%$.

CI: $(\hat{p} - ME, \hat{p} + ME) = (.59 - .03, .59 + .03)$
$= (.56, .62)$

For ME, you could also use the more conservative formula

$ME = z^* \sqrt{\frac{1/4}{n}} = \left(\frac{1.96}{2}\right)\cdot\frac{1}{\sqrt{998}} \approx 0.03$

5.3.20

$\hat{p} = \frac{86}{202}$, $n = 202$

$ME = z^* SE(\hat{p}) = z^* \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = (1.96)\sqrt{\frac{\frac{86}{202}\left(\frac{116}{202}\right)}{202}}$

$\approx (1.96)(0.035) \approx 0.068$

CI: $\left(\frac{86}{202} - 0.068, \frac{86}{202} + 0.068\right)$

$= (0.36, 0.49)$

Since .5 is not in our CI, we doubt that the true proportion is that high. However, since it is very close to our upper bound, we should be cautious in ruling it out. We would probably suggest further study.

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5.4.1

The sample space is

(1,2) (2,3) (3,4)
(1,3) (2,4) (3,5)
(1,4) (2,5) (4,5)
(1,5)

These outcomes are all equally likely, so
$\mathbb{P}(j,k)=\frac{1}{10}$ for all j and k.

Let X_1 and X_2 be the numbers drawn. Then $\hat{\theta}=\frac{X_1+X_2}{2}$.

$$\begin{aligned}\mathbb{P}(|\hat{\theta}-3|>1) &= \mathbb{P}(\hat{\theta}-3>1)+\mathbb{P}(\hat{\theta}-3<-1)\\ &= \mathbb{P}(\hat{\theta}>4)+\mathbb{P}(\hat{\theta}<2)\\ &= \mathbb{P}(X_1+X_2>8)+\mathbb{P}(X_1+X_2<4)\\ &= \mathbb{P}((4,5))+\mathbb{P}((1,2))\\ &= 2/10\end{aligned}$$

5.4.2 $\hat{\theta}=Y_{max}$, $n=6$

$f_Y(y)=1/\theta$, $0\le y\le\theta$

$F_Y(y)=\int_0^y \frac{1}{\theta}\,dt=\frac{y}{\theta}$, $0\le y\le\theta$

$f_{Y_{max}}(y)=n\,[F_Y(y)]^{n-1}f_Y(y)=6\left[\frac{y}{\theta}\right]^5\left(\frac{1}{\theta}\right)$

$= 6y^5/\theta^6$, $0\le y\le\theta$ ← $\theta=3$

$n=6$

(a) $\mathbb{P}(|\hat{\theta}-3|<.2)=\mathbb{P}(-.2<\hat{\theta}-3<.2)=\mathbb{P}(2.8<\hat{\theta}<3.2)$

$=\int_{2.8}^{3.2} f_{Y_{max}}(y)\,dy=\int_{2.8}^{3}\frac{6y^5}{\theta^6}\,dy=\frac{6}{\theta^6}\cdot\frac{y^6}{6}\Big|_{2.8}^{3}$

$=\frac{3^6}{\theta^6}-\frac{2.8^6}{\theta^6}=\frac{3^6}{3^6}-\frac{2.8^6}{3^6}=1-\left(\frac{2.8}{3}\right)^6\approx 0.339$

(b) $n=3$: $\mathbb{P}(|\hat{\theta}-3|<.2)=\int_{2.8}^{3.2} f_{Y_{max}}(y)\,dy=\int_{2.8}^{3} 3\left[\frac{y}{\theta}\right]^2\left(\frac{1}{\theta}\right)dy$

$=\int_{2.8}^{3}\frac{y^2}{3^2}\,dy=\frac{y^3}{3^3}\Big|_{2.8}^{3}=1-\left(\frac{2.8}{3}\right)^3\approx 0.187$

5.4.3 $n=500$, $p=.52$

$\mathbb{P}(\hat{p}<.5)=\mathbb{P}\left(\frac{\hat{p}-p}{SD(\hat{p})}<\frac{.5-.52}{\sqrt{\frac{(.52)(.48)}{500}}}\right)$

$\approx\mathbb{P}(Z<-0.90)\approx 0.1841$

5.4.4 $n=16$, $\sigma=10$, $\hat{\mu}=\bar{Y}$, $\mu=20$

By CLT, $\bar{Y}$ is approx. normal with
$E[\bar{Y}]=\mu$ and $SD(\bar{Y})=\frac{\sigma}{\sqrt{n}}$

In this case,
$E[\bar{Y}]=20$ and $SD(\bar{Y})=\frac{10}{\sqrt{16}}=\frac{10}{4}=\frac{5}{2}$

So

$\mathbb{P}(19<\bar{Y}<21)=\mathbb{P}\left(\frac{19-20}{5/2}\le\frac{\bar{Y}-\mu}{SD(\bar{Y})}\le\frac{21-20}{5/2}\right)$

$=\mathbb{P}(-2/5\le Z\le 2/5)\approx 0.3108$

5.4.9

$f_Y(y;\theta)=2y\theta^2$, $0<y<1/\theta$

$E[c(Y_1+2Y_2)]=c\left(E[Y_1]+2E[Y_2]\right)=3c\,E[Y]$

$=3c\int_0^{1/\theta} y\cdot 2y\theta^2\,dy=6\theta^2c\int_0^{1/\theta}y^2\,dy$

$=6\theta^2c\cdot\frac{y^3}{3}\Big|_0^{1/\theta}=\frac{6\theta^2c}{3}\cdot\left(\frac{1}{\theta}\right)^3=2c\cdot\frac{1}{\theta}$

In order for $E[c(Y_1+2Y_2)]=\frac{1}{\theta}$, we must have $c=1/2$

5.4.10 $n=1$

$f_Y(y;\theta)=\frac{1}{\theta}$ for $0\le y\le\theta$

Note

$E[Y^2]=\int_0^{\theta}y^2\cdot\frac{1}{\theta}\,dy=\frac{1}{\theta}\,\frac{y^3}{3}\Big|_0^{\theta}=\frac{\theta^3}{3\theta}=\frac{\theta^2}{3}$

Thus

$E[3Y^2]=\theta^2$

We conclude that $\hat{\theta}=3Y^2$ is an unbiased estimator for θ^2.