# CS 170 Algorithms Spring 2014 Elchanan Mossel HW 9

## Due April 18th, 6:00pm

**1. (10 pts.) Non-Linear Programming**

A *quadratic programming* problem seeks to maximize a quadratic objective function (with terms like $3x_1^2$ or $5x_1x_2$ ) subject to a set of linear constraints. Give an example of a quadratic program in two variables $x_1$, $x_2$ such that the feasible region is nonempty and bounded, and yet none of the vertices of this region optimize the (quadratic) objective.

**2. (15 pts.) Generalized Max Flow**

Consider the following generalization of the maximum flow problem.

You are given a directed network $G = (V,E)$ with edge capacities $\{c_e\}$. Instead of a single $(s,t)$ pair, you are given multiple pairs $(s_1,t_1),...,(s_k,t_k)$, where the $s_i$ are sources of $G$ and $t_i$ are sinks of $G$. You are also given $k$ demands $d_1,...,d_k$. The goal is to find $k$ flows $f^{(1)},...,f^{(k)}$ with the following properties:

(a) $f^{(i)}$ is a valid flow from $s_i$ to $t_i$.

(b) For each edge $e$, the total flow $f_e^{(1)} + f_e^{(2)} + ... + f_e^{(k)}$ does not exceed the capacity $c_e$.

(c) The size of each flow $f^{(i)}$ is at least the demand $d_i$.

(d) The size of the *total* flow (the sum of the flows) is as large as possible.

How would you solve this problem?

**3. (20 pts.) Sugar Water Enterprises**

Your parents have just left for Costa Rica and have, surprisingly, left you in charge of their beverage company, Sugar Water Enterprises. Unfortunately, they failed to decide on a production plan before they left, and so you must do so now. Your company uses sugar, which costs 10 cents per gram, caffeine, which costs 20 cents per milligram, and water, which costs 1 cent per ounce to produce its beverages. There are two beverages that your company produces, Zap Energy and the signature Sugar Water. One case of Zap Energy is made with 5 grams of sugar, 1 milligram of caffeine and 8 ounces of water. One case of Sugar Water is made with 12 grams of sugar and 16 ounces of water. Your Sugar Water machine can't produce more than 80 Sugar Water cases in a day, and your Zap Energy machine can't produce more than 50 Zap Energy cases in a day. You have a budget of $b$\$ that can be spent on materials. Zap Energy sells for 5\$ per case, and Sugar water sells for 2\$ per case. The goal is to maximize your profit in one day.

(a) Create a linear program to represent this problem.

(b) Find the dual of the linear program.

(c) Find the optimal solution as a function of $b$ (assume you can produce non-integer numbers of cases and can purchase items in non-integer quantities).

**4. (20 pts.) Pizza Battles**

The pizza business in Little Town is split between two rivals, Tony and Joey. They are each investigating strategies to steal business away from the other. Joey is considering either lowering prices or cutting bigger slices. Tony is looking into starting up a line of gourmet pizzas, or offering outdoor seating, or giving free sodas at lunchtime. The effects of these various strategies are summarized in the following payoff matrix (entries are dozens of pizzas, Joey's gain and Tony's loss).

| | | Tony | | |
|---|---|---|---|---|
| | | Gourmet | Seating | Free soda |
| Joey | Lower price | +2 | 0 | -3 |
| | Bigger slices | -1 | -2 | +1 |

For instance, if Joey reduces prices and Tony goes with the gourmet option, then Tony will lose 2 dozen pizzas worth of business to Joey. What is the value of this game, and what are the optimal strategies for Tony and Joey?

**5. (15 pts.) Convex Optimization**

Consider the following convex optimization problem:

$$\max_{x_1,x_2} (ax_1+bx_2+c)^2 \ : x_1 \geq 0,\ x_2 \geq 0$$
$$x_1+x_2 \leq 1$$

(a) First, show that $f(x) = (ax_1+bx_2+c)^2$ is convex, i.e. $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ for all $x,y \in \mathbb{R}^2$ and $0 \leq \lambda \leq 1$. Notice here that $x$ and $y$ here are vectors so $x = (x_1,x_2)$ and $y = (y_1,y_2)$.

(b) Show that the optimal point of the convex optimization problem is achieved at a vertex of the feasible set (set of points for which all constraints are held).