ENGR 105: Feedback Control Design
Winter 2014

# Lecture 10 - Error Response and System Type

Wednesday, January 29, 2014

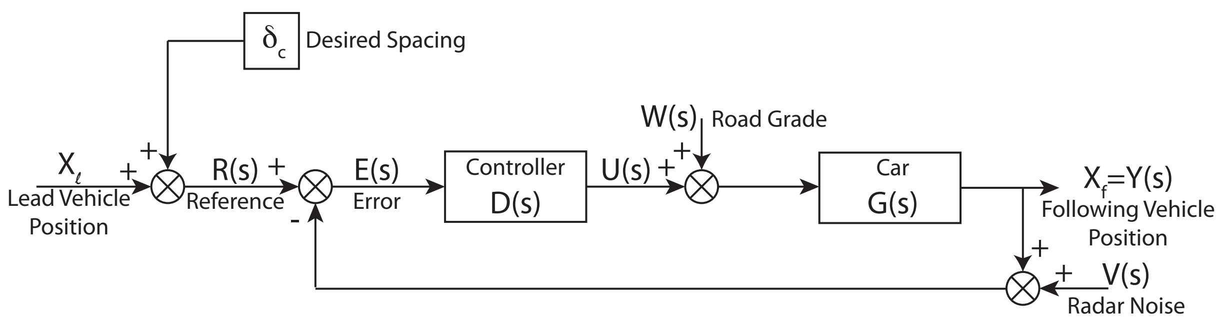
**Today's Objectives**

1. look at error response to common reference inputs
2. define system types in terms of error response
3. introduce integral control and its ability to change system type

Reading: FPE Sections 4.2 and 4.3

## 1 Error Response

Recall our example car-following system:




The error, $e(t)$ or $E(s)$, is extremely important. If this value gets too large, the cars may collide. We therefore want to look not only at the transfer function from reference to output, but also at the transfer function from reference to error:

$$\frac{E(s)}{R(s)} = \frac{1}{1 + D(s)G(s)}$$

For the simple case of proportional control, $D(s) = K$ and plant $G(s) = \frac{1}{ms^2+bs}$, we have:

$$\frac{E(s)}{R(s)} = \frac{1}{1 + \frac{K}{ms^2+bs}} = \frac{ms^2 + bs}{ms^2 + bs + K}$$

Let's assume that we have picked a value of $K$ that makes the system stable. Since the system is stable, we can use the final value theorem to look at the response to different reference inputs:

**Step change:** $R(s) = \frac{1}{s}$

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{ms^2 + bs}{ms^2 + bs + K} = 0$$

$\Rightarrow$ No steady-state error to a step reference

**Ramp:** $R(s) = \frac{1}{s^2}$ (step change in lead vehicle speed)

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{ms + b}{ms^2 + bs + K} = \frac{b}{K}$$

$\Rightarrow$ Finite error to a ramp input. Error reduces as $K$ increases.

**Parabola:** $R(s) = \frac{2}{s^3}$ (step change in lead vehicle acceleration)

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{2(ms + b)}{ms^3 + bs^2 + Ks}$$

$\Rightarrow$ Error grows to infinity!

This is a problem! If the lead vehicle starts to brake, the spacing error continues to grow.

In discussing the steady-state response with regard to a particular input, control engineers use the term "System Type".

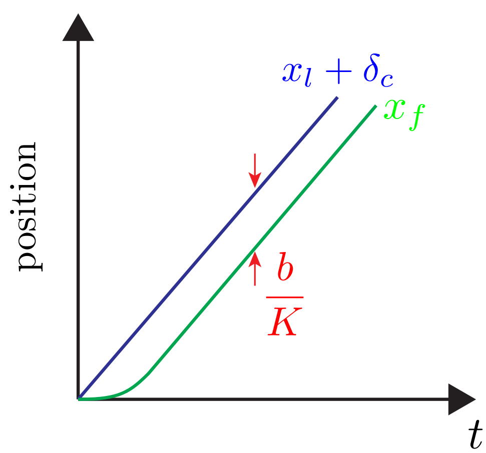
## 2 System Type

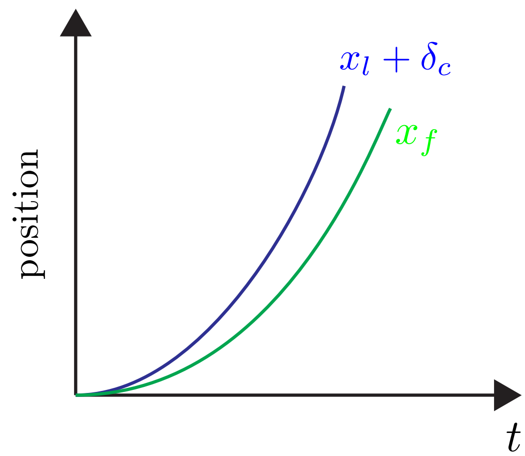
**Type 0:** Finite error to a step (position error)

**Type 1:** Finite error to a ramp (velocity error)

**Type 2:** Finite error to a parabola (acceleration error)

Because our car-following system has a finite error to a reference ramp, it is Type 1 with respect to the reference.




To determine system type for a *unity feedback* connection (no blocks in the feedback loop), you can look at the *loop gain* or *loop transfer function*.

$$L(s) = D(s)G(s) \qquad \text{this is the loop transfer function}$$

$$= \frac{L_0(s)}{s^n} \qquad n \text{ integrators mean a system type of } n$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+L(s)} \qquad \text{reference: } R(s) = \frac{k!}{s^{k+1}} \text{ for } r(t) = t^k$$

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \qquad \text{final value theorem}$$

$$= \lim_{s\to 0} \frac{1}{1+L(s)} \cdot s \cdot \frac{k!}{s^{k+1}}$$

$$= \lim_{s\to 0} \frac{1}{1+\frac{L_0(s)}{s^n}} \cdot \frac{k!}{s^k}$$

$$= \lim_{s\to 0} \frac{s^n}{s^n + L_0(s)} \cdot \frac{k!}{s^k} = 0 \quad \text{if } n > k \Rightarrow \; n \text{ is the system type}$$

To be on the safe side, you can always use the final value theorem to check instead of remembering a particular formula. The key thing to understand is that system type is related to the number of integrators in the loop transfer function – the more integrators, the higher the system type. And thus, the ability to follow higher order reference inputs with finite error. Note that system type is not the same as stability; ramp and higher order reference inputs are not bounded so they are not relevant to our definition of bounded-input bounded-output stability.

In addition to looking at the reference input, it is also possible to characterize the system type with regard to a disturbance. This gives some measure of the disturbance rejection capability of the system.

$$\frac{E(s)}{W(s)} = \frac{-G(s)}{1 + D(s)G(s)} = \frac{-1}{ms^2 + bs + K}$$

$$e_{\mathrm{ss}} = \lim_{s \to 0} s \cdot \frac{-1}{ms^2 + bs + K} \cdot \frac{A}{s} = -\frac{A}{K} \qquad \text{for a step change } A \text{ in road grade}$$

$\Rightarrow$ The system is type 0 with respect to disturbance.

System type does not depend upon the chosen control gains (in this case, $K$). It is instead related to the structure of the plant and controller. If we want to change the system type, we need to change the controller structure.

How can we change this system so that the following vehicle can track an accelerating/decelerating lead vehicle and reject disturbances?

# 3 Integral Control

The current system is Type 1 with regard to reference inputs. We would like to make it into a Type 2 system. In order to do this, we need to add an integrator to $L(s) = D(s)G(s)$.

**PI controller:** $D(s) = K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s}$

This means $U(s) = D(s)E(s)$ or

$$u(t) = K_p e(t) + K_i \int_0^t e(t)\ dt$$

If the initial error is quite large, the integral of the error can be huge, resulting in a large overshoot. This effect is known as *integrator wind-up* and is something to be careful about when using integral control.

With the integral controller, the loop gain is

$$L(s) = \frac{K_p s + K_i}{s} \cdot \frac{1}{s(ms + b)} = \frac{K_p s + K_i}{s^2(ms + b)}$$

↑ two integrators in the denominator

$$\frac{E(s)}{R(s)} = \frac{1}{1+\frac{K_p s + K_i}{s^2(ms+b)}} = \frac{s^2(ms+b)}{ms^3+bs^2+K_p s+K_i}$$

For a **ramp reference**, the steady state error is given by

$$e_{\text{ss}} = \lim_{s\to 0} s \cdot \frac{s^2(ms+b)}{ms^3+bs^2+K_p s+K_i} \cdot \frac{1}{s^2} = 0$$

So the system will have zero error to a ramp. For a parabola (characteristic of an accelerating or decelerating lead vehicle):

$$e_{\text{ss}} = \lim_{s\to 0} s \cdot \frac{s^2(ms+b)}{ms^3+bs^2+K_p s+K_i} \cdot \frac{2}{s^3} = \frac{b}{K_i}$$

$K_i$ can be tuned to give acceptable error so the cars don't collide.

For a **disturbance**:

$$\frac{E(s)}{W(s)} = \frac{\frac{1}{s(ms+b)}}{1+\frac{K_p s + K_I}{s^2(ms+b)}} = \frac{s}{ms^3+bs^2+K_p s+K_i}$$

So for a step change in the road grade:

$$e_{\text{ss}} = \lim_{s\to 0} s \cdot \frac{s}{ms^3+bs^2+K_p s+K_i} \cdot \frac{1}{s} = 0$$

This all looks good, but remember that the final value theorem is only valid for stable systems. Do we have a stable system? The characteristic equation is:

$$1 + D(s)G(s) = 0$$
$$ms^3 + bs^2 + K_p s + K_i = 0$$

Is it enough to say that $m, b, K_p, K_i > 0$? It turns out it isn't. We can always solve for the poles and check if they are in the left half plane, but there is an analytical way to check based on the characteristic equation. We'll examine this tool, the *Routh Array*, in the next lecture.